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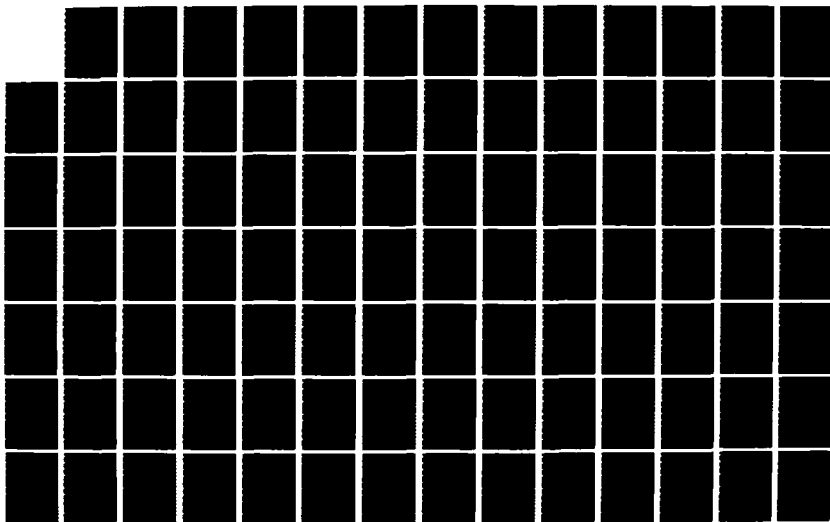
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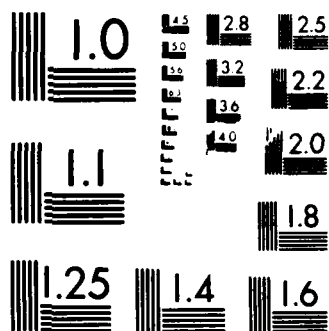
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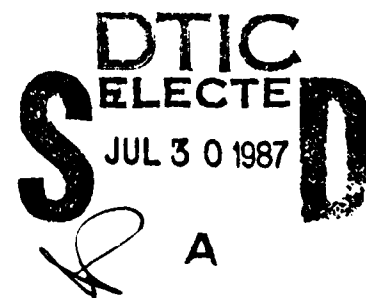
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# ROBUSTNESS IN FEEDBACK SYSTEMS

**Thomas Leo Ting**



**UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN**

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ROBUSTNESS IN FEEDBACK SYSTEMS

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THESIS

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## ROBUSTNESS IN FEEDBACK SYSTEMS

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University of Illinois at Urbana-Champaign, 1987

The research in this dissertation is motivated by the basic question "What can and cannot be accomplished by feedback control?" In particular, this dissertation shall address three basic issues: a) Determining which types of controllers are optimal for certain classes of control problems; b) Investigating the absolute limitations of feedback control for multiobjective problems and the performance tradeoffs available between the various objectives, and c) Developing efficient methods for synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainty.

With regard to the issues identified above, the principal contributions of this dissertation can be outlined as follows. First, it is shown that for the problem of robustly stabilizing a family of plants featuring dynamic uncertainty, linear time-invariant controllers perform as well as arbitrary nonlinear time-varying controllers. Second, a new controller synthesis procedure called residue iteration is developed for synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainty. This method is simpler and numerically more attractive than any previously existing technique. Finally, an algorithm is presented which enables one to compute an absolute upper bound on the performance levels attainable in multiobjective  $H_\infty$ -optimization problems.

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## CHAPTER 1

### INTRODUCTION

#### 1.1. Motivation and Background

The main thrust of control theory since the 1960's has been in the direction of controlling accurately modeled, highly structured systems. However, in many situations available models are partial, inaccurate and far from being clearly structured. These modeling uncertainties arise from a variety of sources such as linearization of nonlinear dynamics, drift of plant parameters during operation, errors due to model reduction techniques, and identification errors. A method of dealing with these modeling uncertainties is to allow the true plant model to be any of a family  $F$  of possible plant models. Any controller designed must then achieve the desired performance specifications not only for the nominal plant model, but rather for the entire family, i.e., the controller must be robust with respect to the modeling uncertainty described by the family  $F$ . In classical control theory the concepts of gain and phase margin are employed to study the robust stability properties of feedback systems (see Horowitz [1963]), but these provide no systematic design methodology for the synthesis of robust feedback controllers.

For single-input, single-output systems the relationship between a system's robustness properties and the return difference matrix is well-known (see Bode [1945] or Horowitz [1963]). A central focus of robust control theory over the years has been an attempt to generalize these results to multiinput, multioutput systems (see for example the works of Cruz and Perkins [1964], Rosenbrock [1974], MacFarlane and Postlethwaite [1977], Safonov and Athans [1977], and Doyle [1979]). Recently, there has been a growing interest in examining robust control problems via frequency domain methods. The appeal of this approach may be linked to several factors:

- a) For many control system design problems, frequency domain plant models are often easier to obtain than state space plant models. Therefore, it is useful to have an alternative design methodology available for problems where state space plant models are difficult to obtain.
- b) Many existing state space design techniques assume perfect knowledge about the plant model and provide no systematic or satisfactory procedure for the treatment of robustness issues. This is at least partially due to the fact that state space models are not well-suited for representing certain types of modeling uncertainty such as dynamic uncertainty (see Chapter 2). On the other hand, in the frequency domain approach this uncertainty can easily be represented by a ball of plant models surrounding a nominal frequency domain plant model. Thus, a treatment of robustness can be readily incorporated into both the problem formulation and the design procedure.
- c) The popular Wiener-Hopf design procedures make restrictive assumptions on the allowable external disturbance inputs. Stochastic disturbance inputs are modeled as random processes with known means and power spectral densities. The resulting Wiener-Hopf design is optimal for this single disturbance input. By contrast, the uniformly (or  $H_\infty$ -) optimal control approach to this problem minimizes the worst-case effect of a set of disturbance inputs and may therefore be more amenable to certain design problems.

In response to these concerns, several new approaches for studying robustness issues have emerged, particularly in the context of multiinput, multioutput systems. These include uniformly (or  $H_\infty$ -) optimal control (see Zames [1981], Zames and Francis [1983], Francis and Zames [1984], and O'Young and Francis [1987]), robustness and stability

margin studies (see Cruz, Freudenberg, and Looze [1981], Doyle and Stein [1981], Lehtomaki et al. [1981], Safonov [1980] and the references cited therein),  $\mu$ -synthesis methods (see Doyle [1984], Doyle, Wall, and Stein [1982] and Safonov and Doyle [1984]), and robust adaptive methods (see Kosut and Friedlander [1985] and Rohrs et al. [1982]).

From a conceptual standpoint the frequency domain approach has introduced some extremely powerful tools for control system design. However, due to the efficiency of state space computational techniques, almost all of the necessary computations involved in synthesizing controllers are still performed using state space techniques. In fact, an important focus of current research is on development of efficient, reliable state space techniques for synthesizing controllers which achieve desired frequency domain performance criteria. This problem is further addressed in Chapter 5.

One primary focus of this thesis is on the so-called robust stabilization problem. For recent research on this problem, see Safonov [1980], Doyle and Stein [1981], Kimura [1984], Vidyasagar [1985], and the references cited therein. This problem is of central importance in robust control theory and may be formulated as follows: Given a family  $F$  of plant models which represents the set of possible "true" plant models (accounting for modeling errors), find (if possible) a single feedback controller  $K$  which stabilizes all plants in the family  $F$ . In complete generality, this problem is extremely difficult and no solution is known. However, for certain special cases of interest a solution may be determined.

For instance, Kimura [1984] has obtained a complete solution to the problem of robustly stabilizing a family of plants described by dynamic, additive uncertainty using linear time-invariant controllers. Also, Tannenbaum [1980] and Khargonekar and

Tannenbaum [1985] have studied the problems of stabilizing one parameter families of plants, including gain and phase margin problems, using linear time-invariant controllers.

In response to realistic modeling problems such as those encountered in decentralized systems, Doyle [1982] has introduced the notion of additive block-structured modeling uncertainty. This leads directly to the problem of robust stabilization of families of plants featuring additive block-structured uncertainty which reduces to a  $\mu$ -synthesis optimization problem. Such problems may be solved iteratively through a method known as  $\Lambda$ -Q iteration (see Doyle [1983]).

While robust stabilization is a key feedback system design attribute, any realistic problem formulation must incorporate performance objectives such as command tracking and disturbance rejection. A particular formulation of this necessity is the uniformly (or  $H_\infty$ -) optimal control problem. Here the objective is to minimize the worst-case effect of a set of frequency shaped disturbances on the output. This problem formulation differs from the popular Wiener-Hopf approach to controller design because it optimizes over a set of possible disturbance inputs whereas the Wiener-Hopf approach optimizes for a single disturbance input. A more complete discussion of this problem is contained in Section 3.4.

Based on this brief review of background material, it is now possible to describe the objective and contribution of this dissertation.

## 1.2. Objective and Contribution of This Dissertation

The research in this dissertation is motivated by the basic question "What can and cannot be accomplished by feedback control?" In particular, this dissertation shall

address three basic issues: a) Determining which types of controllers are optimal for certain classes of control problems, b) Investigating the absolute limitations of feedback control for multiobjective problems and the performance tradeoffs available between the various objectives, and c) Developing efficient methods for synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainty.

In addressing the issues identified above, the principal contributions of this dissertation can be outlined as follows. First, it is shown that for the problem of robustly stabilizing a family of plants featuring dynamic uncertainty, linear time-invariant controllers perform as well as arbitrary nonlinear time-varying controllers. Second, a new controller synthesis procedure called residue iteration is developed for synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainty. This method is simpler and numerically more attractive than any previously existing technique. Finally, an algorithm is presented which enables one to compute an absolute upper bound on the performance levels attainable in multiobjective  $H_\infty$ -optimization problems.

The first issue examined at length is whether or not nonlinear time-varying controllers provide any advantage over linear time-invariant controllers for the problem of robustly stabilizing a family of plants featuring dynamic uncertainty. It is known that for certain robust stabilization problems, such as gain margin problems, nonlinear time-varying controllers are far superior to linear time-invariant ones (see for example Khargonekar and Tannenbaum [1985]). This leads one to believe that nonlinear time-varying controllers may also be useful for other robust stabilization problems. However, it is shown that for the problem of robustly stabilizing a dynamically uncertain family of plant models this is not the case.



In general, adaptive controllers are nonlinear and time-varying, thus, the main result described above demonstrates that adaptive controllers are not useful for robustly stabilizing families of plants exhibiting dynamic uncertainty. Due to the proliferation of controller synthesis methods, results of this type are invaluable in ruling out various synthesis methods for certain classes of control problems.

The above result is not entirely unexpected for the following intuitive reason. Due to the nature of dynamic modeling uncertainty (i.e., possibly unbounded McMillan degree of the true plant model) it is impossible to further identify the true plant via input-output experiments. Hence, a nonlinear time-varying (or adaptive) control strategy will not prove advantageous.

The development preceding this main result rests upon several key intermediate results. The most significant of these results demonstrates that the small-gain theorem is both necessary and sufficient for a certain class of nonlinear operators.

Another issue examined in detail is the development of efficient synthesis techniques for finding robustly stabilizing controllers for a family of plants featuring block-structured uncertainty. At present, the only available technique for solving this problem is the  $\Lambda$ -Q iteration method developed by Doyle [1983]. From a numerical standpoint, this approach is complex because it requires the solution of an infinite dimensional optimization problem. Further, this approach has some drawbacks with regard to the computational difficulties in determining  $\Lambda$  and the subsequent need to find a rational approximation for  $\Lambda$ .

In response to these difficulties, a new controller synthesis procedure is developed in this thesis. This procedure, based on a key observation due to Safonov [1986], requires only the solution of a finite dimensional optimization problem. An added benefit of this

approach is that it yields new insight into the robust stabilization problem. In particular, clean sufficient conditions on the nominal plant  $P_0$  which guarantee infinite additive stability margin are developed.

A crucial step in recovering the controller designed via the residue iteration technique is the determination of an outer interpolating function. A new method of finding such functions is presented. This method yields far lower-order outer functions than those obtained via the methods of Youla et al. [1974] and circumvents the rational approximation problems which arise in the methods of Ball and Helton [1979].

The previous discussion has focused upon a single objective  $H_\infty$ -optimization problem, the robust stabilization of a family of plants. However, many interesting control problems reduce instead to multiobjective  $H_\infty$ -optimization problems. An example of this is the problem of robust stabilization of a family of plants with optimal performance for the nominal plant. These multiobjective problems are extremely difficult to solve analytically. Therefore, the development in this thesis centers on the presentation of a systematic algorithm to establish absolute bounds on the attainable performance levels. These bounds are established graphically and the resulting curves are useful for demonstrating or comparing the tradeoffs inherent between various competing objectives.

### 1.3. Organization of This Dissertation

The remainder of this dissertation is organized as follows. Chapter 2 describes in detail various paradigms for treating the problem of modeling uncertainty. Following this in Chapter 3 a review of key background material which forms the foundation of

work in this dissertation is presented. The relative merits of nonlinear time-varying feedback as compared with linear time-invariant feedback in the context of certain robust stabilization problems are investigated in Chapter 4. Following this, Chapter 5 concentrates on the problem of synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainty. Finally, Chapter 6 describes a systematic algorithm for establishing absolute bounds on the attainable performance levels in multiobjective  $H_\infty$ -optimization problems.

The main results and implications of this dissertation as well as some possible directions for further research are summarized in the final chapter.

#### 1.4. Notation

$\mathbb{R} =$	field of real numbers
$\mathbb{R}^+ =$	set of positive real numbers
$\mathbb{C} =$	field of complex numbers
$\mathbb{C}^* =$	field of complex numbers - $\{0\}$
$D =$	the open unit disk = $\{z \in \mathbb{C} :  z  < 1\}$
$e^{j\theta} =$	the unit circle = $\{z \in \mathbb{C} :  z  = 1\}$
$H =$	open right half plane = $\{s \in \mathbb{C} : \text{Re } s > 0\}$
$\bar{H} =$	closed right half plane = $\{s \in \mathbb{C} : \text{Re } s \geq 0\}$
LTI =	linear time-invariant
LTV =	linear time-varying

NLTV =	nonlinear time-varying
$(\ )^*$ =	adjoint of
$(\ )_+$ =	unstable part of
$\underline{\lambda}(M)$ =	smallest eigenvalue of $M$
$\underline{\sigma}(G)$ =	smallest nonzero singular value of an operator $G = \underline{\lambda}^{1/2}(G^*G)$
$\pi_1$ =	projection from $l_2 \rightarrow h_2$ = causal part of
$\pi_2$ =	projection from $l_2 \rightarrow h_2^\perp$ = anticausal part of
$L_\infty(j\mathbb{R})$ =	Banach space of essentially bounded functions on $j\mathbb{R}$
$H_\infty(j\mathbb{R})$ =	Banach space of essentially bounded functions on $j\mathbb{R}$ which admit an analytic extension to $H$
$RH_\infty(j\mathbb{R})$ =	the space of rational $H_\infty(j\mathbb{R})$ functions
$\  \ \ _H$ =	Hankel norm
$\  \ \ _\infty$ =	$L_\infty$ or $H_\infty$ -norm

For  $g$  in  $L_\infty$  or  $H_\infty$ ,  $\|g\|_\infty = \text{ess sup}_\omega \overline{\sigma}(g(j\omega))$ .

See Rudin [1966] and Glover [1984] for details concerning these definitions.

## CHAPTER 2

### MODELING UNCERTAINTY

Since models are inherently inaccurate, key tasks for the control system designer are to analyze and reduce the effects of modeling uncertainty on the overall system performance. A reasonable formulation of this problem is to represent the true physical plant, not by a single nominal plant, but instead by a family  $F$  of possible plant models. This representation of modeling uncertainty falls into three broad categories.

(a) **Parametric Uncertainty.** In this case  $F$  consists of a continuously parameterized family of LTI plant models as

$$F = \{P_\alpha(z) : \alpha \in \Omega \subseteq \mathbb{R}^n\} \quad (2.1)$$

where the parameter vector  $\alpha$  takes on values in some compact set  $\Omega$ . Note that this description of modeling uncertainty reflects considerable *a priori* knowledge about the plant to be controlled. This is because implicit in (2.1) is the presumption that the designer has been able to isolate the sources of modeling errors in the plant model to specific elements such as gains or pole/zero locations and to conservatively estimate their ranges by compact sets. In many cases adaptive controllers are best suited for treating this type of uncertainty (see for example Astrom [1983], Kumar [1985] and the references cited therein).

(b) **Dynamic Uncertainty.** This type of modeling uncertainty is described by a frequency weighted ball of perturbations around a nominal plant model. This representation reflects considerably less *a priori* knowledge about the physical plant (as compared to parametric uncertainty) and is characterized by a complete lack of knowledge of the McMillan degree of the true plant. A specific instance of dynamic

uncertainty is the additive unstructured uncertainty as in Doyle and Stein [1981] described by

$$F_\delta = \{P_o(z) + \Delta W_2(z) : \|\Delta\| \leq \delta\}. \quad (2.2)$$

Here,  $P_o$  is a nominal LTI plant model,  $\Delta$  is a normalized perturbation and  $W_2(z)$  is a stable and stably invertible frequency-dependent weighting function which characterizes the relative magnitude of modeling uncertainty at various frequency levels. Typically,  $W_2(z)$  takes on larger values at higher frequencies reflecting the inability to model accurately at these frequencies. In addition, it is required that all plant models in  $F_\delta$  have the same number of unstable poles (if  $\Delta$  is LTI) or that  $\Delta$  be a stable NLTV perturbation.

In many problems of practical interest, the representation (2.2) of modeling uncertainty is too conservative. Motivated by this, Doyle [1982] has introduced the notion of block-structured uncertainty wherein only certain subsystems of the plant model exhibit significant modeling errors. Through judicious block-diagram manipulations it can be shown (see Doyle [1982]) that many arbitrary uncertainty structures can be transformed to one with additive block-structured uncertainty. Thus, one may focus on additive block-structured uncertainty to be represented as

$$F_\delta = \left\{ P_o(z) + \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \\ & & & \Delta_n \end{bmatrix} W(z) : \|\Delta_i\| \leq \delta \right\}. \quad (2.3)$$

Here the  $j$ th uncertainty block  $\Delta_j$  is of size  $r_j \times r_j$  and the assumptions of (2.2) apply.

(c) **Mixed Uncertainty.** This representation of modeling uncertainty involves a combination of both parametric and dynamic uncertainties. A paradigm for this

description is

$$F_{\delta} = \{P_{\alpha}(z) + \Delta W_{\alpha}(z) : \|\Delta\| \leq \delta, \alpha \in \Omega \subseteq \mathbb{R}^n\} \quad (2.4)$$

and clearly represents the most realistic situation. In this case, the accurately modeled subsystems of the plant model are described by a parametric representation and the poorly modeled subsystems of the plant model (e.g., the high frequency dynamics) are described by dynamic uncertainty. Recent research articles (see for example, Rohrs et al. [1982]) on the effects of unmodeled high frequency dynamics on adaptive control algorithms address this type of modeling uncertainty.

## CHAPTER 3

## DEFINITIONS AND BACKGROUND MATERIAL

This chapter outlines a mathematical framework which forms the basis for the frequency domain approach to control system analysis and design. Many fundamental concepts of frequency domain control are reviewed and these form the building blocks for much of the further development in this thesis. While only discrete time versions of these ideas are described, continuous time counterparts of these notions may readily be obtained.

This chapter is organized as follows. In Section 3.1 basic input-output theory is discussed, focusing principally on the operator-theoretic representation of systems. Section 3.2 treats stability theory and presents a well-known parameterization of all stabilizing controllers, along with the basic elements of stable coprime and inner-outer factorization theory. Finally, in Sections 3.3 and 3.4 two principal problems in frequency domain control are introduced: the robust stabilization problem and the uniformly (or  $H_\infty$ -) optimal control problem. These problems shall occupy much of the attention of this dissertation.

## 3.1. Input-Output Theory

Throughout this dissertation, the following notation from Willems [1971] shall be employed. Let  $Z_+$  be the set of positive integers and define the usual Hilbert space

$$l_2 = \{u = (\dots u_{-1}, u_0, u_1, \dots) : u_i \in \mathbb{R}, \sum_{i=-\infty}^{\infty} u_i^2 < \infty\}$$

equipped with the usual norm  $\|u\|_2 = (\sum_{i=-\infty}^{\infty} u_i^2)^{1/2}$ . As is well-known,  $l_2$  admits a direct-



sum decomposition as  $l_2 = h_2 + h_2^\perp$  where

$$h_2 = \{u = (u_0, u_1, u_2, \dots) : u_i \in \mathbb{R}, \sum_{i=0}^{\infty} u_i^2 < \infty\}.$$

Let  $bh_2$  be the unit ball in  $h_2$ . Define a projection operator  $\pi_N$  acting on the space of all two-sided sequences (not necessarily square-summable) of real numbers as

$$\pi_N : (\dots u_{-1}, u_0, u_1, \dots) \rightarrow (\dots u_{-1}, u_0, u_1, \dots, u_{N-1}, 0, \dots).$$

As in Willems [1971] let

$$h_{2e} = \{u = (u_0, u_1, u_2, \dots) : \pi_N u \in h_2 \text{ for all } N \in \mathbb{Z}_+\},$$

i.e.,  $h_{2e}$  is the space of all locally  $h_2$  sequences. Similarly the space  $l_{2e}$  can be defined to be the space of all locally  $l_2$  sequences of real numbers.

An  $m$ -input,  $p$ -output nonlinear time-varying (NLTV) operator  $F : h_{2e}^m \rightarrow l_{2e}^p$  will be called proper (causal) if  $\pi_N(u) = \pi_N(v)$  implies that  $\pi_N F(u) = \pi_N F(v)$  for all  $u, v$  in  $h_{2e}^m$  and all  $N$  in  $\mathbb{Z}$ . Note that for a causal operator  $F$ ,  $\text{Im}(F) \subseteq h_{2e}^p$ . Also,  $F$  will be called strictly proper if  $\pi_N(u) = \pi_N(v)$  implies that  $\pi_{N+1} F(u) = \pi_{N+1} F(v)$  for all  $u, v \in h_{2e}^m$  and all  $N$  in  $\mathbb{Z}$ . The norm of  $F$  is defined as

$$\|F\| = \sup_{\substack{u \in h_{2e}^m \\ u \neq 0}} \frac{\|F(u)\|_2}{\|u\|_2}, \quad (3.1)$$

and  $F$  is said to be  $h_2$ -stable (henceforth, stable) if  $\|F\| < \infty$ . In addition, define  $\|F\|_{bh_2}$  to be the norm of  $F$  restricted to inputs  $u$  in  $bh_2$ . Finally, the incremental norm of the operator  $F$  is defined as

$$\|F\|_{inc} = \sup_{\substack{u, v \in h_2^m \\ u \neq 0}} \frac{\|F(u+v) - F(v)\|_2}{\|v\|_2} \quad (3.2)$$

and  $F$  will be called incrementally stable if  $\|F\|_{inc} < \infty$ .

If  $F$  is a linear time-invariant (LTI) operator, then it may be associated with a transfer function matrix  $F(z)$ . In this case  $F$  is stable if and only if  $F(z)$  belongs to  $H_\infty$ , and it can be shown that (see Francis [1987]):

$$\|F\| = \|F(z)\|_\infty = \text{ess sup}_{0 \leq \theta \leq 2\pi} |F(e^{j\theta})| \quad (3.3)$$

A transfer function matrix  $F(z)$  will be called unimodular if and only if  $F(z) \in H_\infty$  and  $F^{-1}(z)$  exists and belongs to  $H_\infty$ .

The orthogonal projection from  $l_2^p$  onto  $h_2^p$  will be denoted  $\pi_1$  and from  $l_2^p$  onto  $(h_2^p)^\perp$  by  $\pi_2$ . Then, with any NLTV operator  $F$  one can associate its Hankel operator  $\Gamma(F)$  and its Toeplitz operator  $\theta(F)$  defined as

$$\Gamma(F) : h_{2e}^m \rightarrow (h_{2e}^p)^\perp : u \rightarrow \pi_2 F u$$

$$\theta(F) : h_{2e}^m \rightarrow h_{2e}^p : u \rightarrow \pi_1 F u.$$

This definition is consistent with the standard time domain interpretation of a Hankel operator as mapping future inputs to past outputs, i.e., the Hankel operator defined from the anticausal projection of  $F$ . The Hankel norm of a NLTV operator  $F$ , denoted by  $\|F\|_H$ , is defined as

$$\|F\|_H = \|\Gamma(F)\| \quad (3.4)$$

If the operator  $F$  is time-invariant it is easy to see that its Hankel norm may be characterized alternatively as

$$\|F\|_H = \lim_{N \rightarrow \infty} \|\pi_N F(I - \pi_N)\| . \quad (3.5)$$

For a finite dimensional LTI operator  $F$ , its associated Hankel operator  $\Gamma(F)$  is of finite rank and, consequently, has finitely many singular values. These are called the Hankel singular values associated with  $F$ , and are fundamental input-output invariants of a system. They capture the complexity of a system model vis-a-vis various feedback problems, particularly in the context of model approximation. It is clear from (3.4) that  $\|F\|_H$  is simply the largest Hankel singular value of  $F$ . Given a purely anticausal transfer function matrix  $F(z)$ , its Hankel singular values may be computed as follows (see Moore [1978]). Let  $\Sigma(A, B, C, D)$  be a canonical realization of  $F(z)$ . Solve the Lyapunov equations

$$A M A^T - M = B B^T \quad (3.6a)$$

$$A^T W A - W = C^T C \quad (3.6b)$$

for the controllability and observability grammians  $M$  and  $W$ , respectively. Then the Hankel singular values of  $F(z)$ , denoted by  $\sigma_i(F)$ , are given by

$$\sigma_i(F) = \lambda_i^{1/2}(MW), \quad i = 1, 2, \dots, N$$

where  $N$  is the McMillan degree of  $F(z)$ .

The central notion of an adjoint operator is now defined.

**Definition (3.1).** Let  $H_1$  and  $H_2$  be Hilbert spaces. Let  $T: H_1 \rightarrow H_2$  be a linear operator. The adjoint of  $T$  written  $T^*$  is the (it can be shown that  $T^*$  always exists and is unique) operator  $T^*: H_2 \rightarrow H_1$  defined by

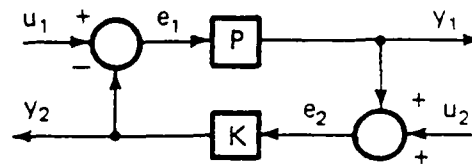
$$\langle y, Tu \rangle_{H_2} = \langle T^* y, u \rangle_{H_1} \quad \forall u \in H_1, y \in H_2 .$$

If  $T$  is a linear time-invariant operator with transfer function matrix  $T(z)$ , then its adjoint  $T^*(z)$  has the transfer function matrix  $T^T(\frac{1}{z})$ .

As is well known, the set of all NLTV causal operators forms a left-distributive algebra with respect to operator addition and composition. This algebra, however, is **not** right-distributive, i.e.,  $A(B+C) \neq AB + AC$  in general, and this is a principal technical obstruction in dealing with NLTV operators.

### 3.2 Stabilizability and Factorization Theory

Consider the feedback system shown in Figure 3.1 where  $P$  and  $K$  are causal NLTV operators.



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Figure 3.1. A standard feedback system.

Following Desoer et al.[1980] and Willems [1971] one has the following

**Definition (3.2).** Consider the NLTV operator

$$\phi(P,K) : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The feedback system Figure 3.1 is said to be well-posed if  $\phi$  is single-valued and causal.

Further, a controller  $K$  (possibly NLTV) is said to (incrementally) stabilize the plant  $P$  if the operator  $\phi(P,K)$  is (incrementally) stable.

**Remark (3.1).** For a linear plant and controller pair the operator  $\phi(P,K)$  can be expressed in its usual  $2 \times 2$  block matrix form

$$\phi(P,K) = \begin{bmatrix} (I+KP)^{-1} & -K(I+PK)^{-1} \\ P(I+KP)^{-1} & (I+PK)^{-1} \end{bmatrix}.$$

However, due to the lack of right-distributivity of nonlinear operators, if  $P$  or  $K$  is nonlinear then  $\phi(P,K)$  cannot be expressed in this form. In this case it is easy to see that if one of the operators

$$\phi_1 = (I+KP)^{-1} \quad \phi_3 = (I-P(-K))^{-1}$$

$$\phi_2 = P(I+KP)^{-1} \quad \phi_4 = -K(I-P(-K))^{-1}$$

is unstable, then  $\phi(P,K)$  is unstable.

□

Let  $P_0(z)$  be any proper LTI plant. Then  $P_0$  admits LTI stable coprime factorizations as

$$P_0 = ND^{-1} = \tilde{D}^{-1} \tilde{N} \tag{3.7}$$

$$XN + YD = 1, \quad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = 1$$

where  $N, \tilde{N}, D, \tilde{D}, X, \tilde{X}, Y,$  and  $\tilde{Y}$  are all stable, proper LTI operators. A given plant  $P_0$  admits many similar factorizations. However, all of these factorizations are unimodularly related. More precisely, the following result may now be obtained.

**Lemma (3.1).** Let  $P_0(z)$  be a proper LTI plant with a LTI stable coprime factorization as

in (3.7). Further suppose  $P_0$  admits an alternate (possibly NLTV) stable coprime factorization as

$$P_0 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1 \quad \begin{array}{l} X_1 N_1 + Y_1 D_1 = I \\ \tilde{N}_1 \tilde{X}_1 + \tilde{D}_1 \tilde{Y}_1 = I \end{array}$$

Then,

$$\begin{pmatrix} N & \tilde{X} \\ D & \tilde{Y} \end{pmatrix} = \begin{pmatrix} N_1 & \tilde{X}_1 \\ D_1 & \tilde{Y}_1 \end{pmatrix} \begin{pmatrix} U & O \\ O & V^{-1} \end{pmatrix}$$

$$\begin{pmatrix} X & Y \\ \tilde{N} & \tilde{D} \end{pmatrix} = \begin{pmatrix} U^{-1} & O \\ O & V \end{pmatrix} \begin{pmatrix} X_1 & Y_1 \\ \tilde{N}_1 & \tilde{D}_1 \end{pmatrix}$$

where  $U$  and  $V$  are unimodular. In other words, all stable coprime factorizations of a given LTI plant are unimodularly equivalent to any LTI stable coprime factorization.

**Proof.** Choose  $U = D_1^{-1} D$ . Then  $D_1 U = D_1 (D_1^{-1} D) = D$  and  $N_1 U = N_1 D_1^{-1} D = N$ . Thus, it need only be shown that  $U = D_1^{-1} D$  is unimodular. Starting with the identity  $X_1 N_1 + Y_1 D_1 = I$ , it readily follows that

$$X_1 N_1 U + Y_1 D_1 U = X_1 N_1 D_1^{-1} D + Y_1 D = X_1 N + Y_1 D = U$$

which implies that  $U \in H_\infty$ . Similarly,  $U^{-1} = X N_1 + Y D_1$  and  $U^{-1} \in H_\infty$ . Therefore,  $U$  is indeed unimodular. Choosing  $V = \tilde{N} \tilde{N}_1^{-1}$  and employing an identical argument complete the proof. □

The following theorem represents a very special case of a general result due to Ananthram and Desoer [1984]. Their results give a complete global parameterization of

all NLTV stabilizing controllers for the class of strongly stabilizable nonlinear plants.

Their results may be specialized to LTI plants to yield the following:

**Theorem (3.1).** Let  $P_0(z)$  be a strictly proper linear time-invariant plant. Then the set of all NLTV controllers that stabilize  $P_0$  is completely parameterized by

$$K = (\tilde{X} + DQ)(\tilde{Y} - NQ)^{-1} \quad (3.8)$$

where the Youla parameter  $Q$  is allowed to be any possibly NLTV stable, causal operator.

**Proof.** From Khargonekar et al. [1985] it follows that any LTI plant  $P_0$  can be stabilized by a stable, periodic linear controller  $K_0$ . Then  $P_0$  admits the following stable coprime factorization

$$P_0 = ND^{-1} = \tilde{D}^{-1} \tilde{N}$$

with

$$XN + YD = I, \quad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I$$

where

$$N = P_0(I + K_0P_0)^{-1}, \quad D = (I + K_0P_0)^{-1}, \quad X = K_0, \quad Y = I$$

$$\tilde{N} = (I + P_0K_0)^{-1}P_0, \quad \tilde{D} = (I + P_0K_0)^{-1}, \quad \tilde{X} = K_0 \text{ and } \tilde{Y} = I.$$

Using results from Ananthram and Desoer [1984], the set of all NLTV controllers  $K$  which stabilize  $P_0$  can be parameterized as

$$K = K_0 + Q(I - P_1Q)^{-1} \quad \text{where } P_1 = N = P_0(I + K_0P_0)^{-1} \quad (3.9)$$

and  $Q$  is allowed to be any stable, causal NLTV operator. This is now simplified as

$$\begin{aligned} K &= K_0 + Q(I - NQ)^{-1} = [K_0(I - NQ) + Q](I - NQ)^{-1} \\ &= [K_0 + (I - K_0N)Q](I - NQ)^{-1} = (K_0 + DQ)(I - NQ)^{-1} \end{aligned} \quad (3.10)$$

$$= (\tilde{X} + DQ)(\tilde{Y} - NQ)^{-1}.$$

Now assume that  $P_0$  has an alternate stable coprime factorization:

$$P_0 = N_1 D_1^{-1} = \tilde{D}_1^{-1} \tilde{N}_1$$

with

$$X_1 N_1 + Y_1 D_1 = I, \quad \tilde{N}_1 \tilde{X}_1 + \tilde{D}_1 \tilde{Y}_1 = I. \quad (3.11)$$

Employing this factorization the set of all controllers that stabilize  $P_0$  are parameterized as

$$K_1 = (\tilde{X}_1 + D_1 Q_1)(\tilde{Y}_1 - N_1 Q_1)^{-1} \quad (3.12)$$

where  $Q_1$  is allowed to be any stable, causal NLTV operator. However, from Lemma (3.1) it follows that all coprime factorizations of a strictly proper LTI plant are unimodularly related. Thus, without loss of generality  $\tilde{X}_1$ ,  $D_1$ ,  $\tilde{Y}_1$ , and  $N_1$  may be chosen from any LTI stable, coprime factorization of  $P_0$  completing the proof. □

**Remark (3.2).** The set of all LTI controllers  $K$  that stabilize  $P_0$  may be parameterized by (3.8) by restricting  $Q$  to be any LTI stable, causal operator. Also, observe that a controller  $K$  incrementally stabilizes  $P_0$  if and only if the corresponding Youla parameter  $Q$  is incrementally stable. Further from (3.7) and (3.8) it follows that

$$(I + P_0 K)^{-1} = (\tilde{Y} - NQ) \tilde{D} \quad (3.13a)$$

$$K(I + P_0 K)^{-1} = (\tilde{X} + DQ) \tilde{D} \quad (3.13b)$$

$$P_0 K(I + P_0 K)^{-1} = N(X + Q\tilde{D}). \quad (3.13c)$$

□

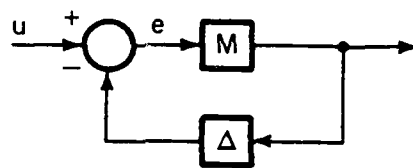
It is significant to note that all three of these transfer functions are affine in  $Q$ , i.e., they



all have the general form  $A+BQC$  where  $A$ ,  $B$ , and  $C$  are fixed stable proper transfer functions (see also (3.21)).

□

Consider the general feedback system shown in Figure 3.2.



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Figure 3.2. Feedback system for the small-gain theorem.

A central theorem due to Sandberg [1964] and Zames [1966] concerning the stability of this system is the following

**Theorem (3.2).** (The Small-Gain Theorem). Consider the feedback system shown in Figure 3.2 and assume that  $M$  and  $\Delta$  are causal operators. If  $\|M\| \leq 1$  then the feedback system is stable for all operators  $\Delta$  with  $\|\Delta\| < 1$ .

**Remark (3.3).** It is crucial to note that the small-gain theorem provides only a **sufficient** condition for the stability of a feedback system. For certain special cases (i.e., when  $M$  and  $\Delta$  are both LTI operators), it is known that the small-gain theorem is both a necessary and sufficient condition for system stability. Indeed, a key result of this dissertation is that the small-gain theorem is both **necessary** and **sufficient** for stabilizing a system consisting of a particular class of NLTV operators.

□

The powerful tools of inner-outer factorizations are now introduced. For a more detailed discussion refer to Duren [1970] or Francis [1987]. A matrix transfer function  $F$  in  $RH_\infty$  is **inner** if  $F^*F = I$ . Inner transfer functions are all-pass. In order for  $F$  to be inner, clearly  $F$  must be tall (number of rows  $\geq$  number of columns). An important property of inner functions is that they preserve inner products under left-multiplication, i.e., for any  $G \in L_\infty$ ,

$$\|FG\|_\infty = \|G\|_\infty.$$

A matrix transfer function  $F$  in  $RH_\infty$  is **outer** if for every  $z$  with  $|z| < 1$ ,  $F(z)$  has full row rank, i.e.,  $F$  has a right-inverse which is analytic in  $|z| < 1$ . Outer functions must be wide (number of columns  $\geq$  number of rows).

Every matrix transfer function  $F \in RH_\infty$  admits an inner-outer factorization as

$$F = F_i F_o \quad (3.14)$$

where  $F_i$  is inner and  $F_o$  is outer. These factorizations may be computed using spectral factorization theory and involve the solution of a set of algebraic Riccati equations (see Doyle [1983] or Francis [1987]).

In addition, a matrix transfer function  $F$  in  $RH_\infty$  is said to be **co-inner** or **co-outer** if  $F^T$  is inner or outer, respectively. Every matrix transfer function  $F \in RH_\infty$  admits a co-outer-co-inner factorization as

$$F = F_{co} F_{ci} \quad (3.15)$$

where  $F_{co}$  is co-outer and  $F_{ci}$  is co-inner. These factorizations are obtained from an inner-outer factorization of  $F^T$ . An important property of co-inner functions is that they preserve inner products under right-multiplication, i.e., for any  $G \in L_\infty$ ,

$$\|GF\|_{\infty} = \|G\|_{\infty}.$$

### 3.3. The Robust Stabilization Problem

As described earlier, plant modeling uncertainty may be treated by allowing the true plant model  $P$  to be any of a family  $F_{\delta}$  of possible plant models. Thus, instead of stabilizing a single plant, a controller must stabilize an entire family of plant models. This leads naturally to the robust stabilization problem which may be formulated as follows: Given a family of plants  $F_{\delta}$  find (if possible) a single controller  $K$  which stabilizes every plant in the family  $F_{\delta}$ . If such a controller exists, then it is called a robustly stabilizing controller and the family of plants is called robustly stabilizable.

The robust stabilization problem has been extensively studied for families of plants  $F_{\delta}$  featuring unstructured uncertainty as described in (2.2) (see for example Doyle et al. [1982], Doyle and Stein [1981], Glover [1984], Kimura [1984], and Verma [1985]). Kimura [1984] has obtained necessary and sufficient conditions for robust stabilizability of  $F_{\delta}$  using LTI controllers based on classical Nevanlinna-Pick interpolation theory. Of particular relevance to this dissertation is the following result due to Doyle and Stein [1981] and Chen and Desoer [1982].

**Theorem (3.3).** Consider a family of plants  $F_{\delta}$  as described in (2.2) and introduce stable coprime factorizations of  $P_0$  as in (3.7). Then the following are equivalent:

(a) There exists a LTI controller  $K$  that robustly stabilizes  $F_{\delta}$ .

(b) 
$$\inf_{\text{LTI } K \text{ stabilizing } P_0} \|W_2 K (I + P_0 K)^{-1}\|_{\infty} < \frac{1}{\delta}.$$

$$(c) \inf_{Q \in H_\infty} \|W_2 \tilde{X} \tilde{D} + W_2 D Q \tilde{D}\|_\infty < \frac{1}{\delta}.$$

□

The corresponding robustly stabilizing LTI controllers may be readily computed using a variety of methods (see for example Vidyasagar [1985]).

Consider again the family of plant models  $F_\delta$  and define  $P_u = (P_0 W_2^{-1})_+$  and  $P_s = P_0 W_2^{-1} - P_u$ . Let  $N$  = McMillan degree of  $P_u$ . Working independently Glover [1986] and Verma [1985] have obtained the following alternate characterization of robust stabilizability of the family  $F_\delta$ :

**Theorem (3.4):** There exists a LTI controller  $K$  that robustly stabilizes the family  $F_\delta$  if and only if

$$\underline{\sigma}(\Gamma(P_0 W_2^{-1})) > \delta. \quad (3.16)$$

In this event, a robustly stabilizing  $K$  may be determined by

$$K = -W_2^{-1} (P_{N-1} + P_s)^{-1}$$

where  $P_{N-1}$  is any  $(N-1)$ st order optimal Hankel norm approximant of  $P_u$  such that  $P_u - P_{N-1}$  is all-pass, i.e.,

$$\|P_u - P_{N-1}\|_\infty = \|\Gamma(P_u - P_{N-1})\| = \underline{\sigma}(\Gamma(P_0 W_2^{-1})).$$

**Remark (3.4).** It is important to note (see Glover [1986] and Verma [1985]) that Theorem (3.4) holds with the relaxed assumption that all plants in  $F_\delta$  have only the same or fewer poles as the nominal plant  $P_0$ . However, applying some basic results from model approximation theory (see Glover [1986]), it readily follows that even under this relaxed assumption the family  $F_\delta$  is robustly stabilizable if and only if all plants in  $F_\delta$  have the same number of unstable poles. Therefore, Theorems (3.3) and (3.4) are indeed

interchangeable robust stability criteria for families of plants  $F_\delta$  as described by (2.2).

Motivated by these results and by more general synthesis problems such as robust performance, Doyle [1982] has introduced notion of a **structured singular value**  $\mu$  which provides a method of tackling the robust stabilization problem for families of plants featuring block-structured uncertainties. The theory developed in Doyle [1982] is particularly clean for the case of three or fewer blocks ( $n \leq 3$ ) where an exact determination of  $\mu$  is possible via the  $\Lambda$ -scaling iteration (see also work by Fan and Tits [1986]). The following development concentrates on this case ( $n \leq 3$ ) and for the case of more than three blocks, the subsequent results yield conservative estimates of robust stability margins.

Following Doyle [1982] define the set

$$\Omega = \left\{ \begin{bmatrix} d_1 I_{j_1} & & 0 \\ & d_2 I_{j_2} & \\ 0 & & d_n I_{j_n} \end{bmatrix} : d_j, d_j^{-1} \in H_\infty \right\} \quad (3.17)$$

where  $I_j$  is the  $j \times j$  identity matrix. The following result, analogous to Theorem (3.3), is easily obtained.

**Theorem (3.5).** Consider the family of plants  $F_\delta$  as described in (2.3) with block-structured uncertainty. Introduce stably coprime factorizations of  $P_0(z)$  as in (3.7). Then the following are equivalent:

(a) There exists a LTI controller  $K$  that robustly stabilizes  $F_\delta$ .

(b) 
$$\inf_{\substack{K \text{ stabilizing } P_0 \\ \Lambda \in \Omega}} \|\Lambda W_2 K (I + P_0 K)^{-1} \Lambda^{-1}\|_\infty < \frac{1}{\delta}.$$

$$(c) \inf_{\substack{Q \in H_{\infty} \\ \Lambda \in \mathcal{C}}} \|\Lambda(W_2\tilde{X}\tilde{D} + W_2DQ\tilde{D})\Lambda^{-1}\|_{\infty} < \frac{1}{\delta}. \quad (3.18)$$

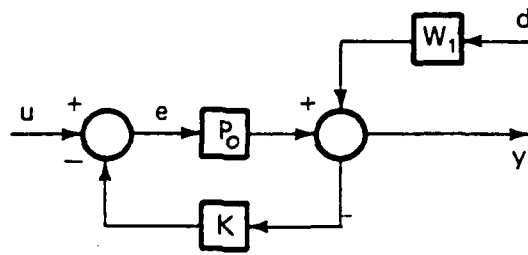
□

An approach to solving the optimization problem suggested by (3.18) of the above theorem is the  $\Lambda$ -Q iteration (or  $\mu$ -synthesis) method of Doyle [1983] and Doyle, Wall, and Stein [1982]. Essentially this method proposes computing the infimum in (3.18) iteratively by freezing  $\Lambda$ , computing the optimal  $Q$ , freezing  $Q$ , and finding the best new scaling  $\Lambda$  and so on. However, although the optimization problem (3.18) is convex in both  $Q$  and  $\Lambda$  it is not jointly convex (see Doyle [1983]). Therefore, there is no guarantee that the  $\Lambda$ -Q iteration method will converge to a global minimum. In addition, the  $\Lambda$ -Q iteration method is numerically unappealing due to the complexity of computing  $\Lambda$  and then having to find a suitable rational approximant for  $\Lambda$  as required during each step of the iteration.

In Chapter 5 a key observation due to Safonov [1986] is employed to develop an alternative to  $\Lambda$ -Q iteration for finding robustly stabilizing controllers for families of plants featuring block structured uncertainty. This alternative procedure, called residue iteration, is numerically attractive in comparison with  $\Lambda$ -Q iteration because it replaces an infinite dimensional optimization problem with a finite dimensional optimization problem. Furthermore, residue iteration is relatively easy to implement numerically, and circumvents the numerical difficulties involved with computing  $\Lambda$ .

### 3.4. Uniformly or $H_\infty$ -Optimal Control

In addition to feedback system stability, in practical control problems it is imperative to incorporate some performance objectives such as disturbance rejection or reference command tracking. This objective falls naturally within the framework of the following uniformly (or  $H_\infty$ -) optimal control problem. Consider the system shown in Figure 3.3.



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Figure 3.3. Feedback system for the uniformly optimal control problem.

Here,  $P_0$  represents a LTI plant model,  $d$  is a disturbance input of unit energy and  $W_1$  is a coloring filter (weighting function) which shapes the frequency characteristics of the disturbance input. Without loss of generality  $W_1$  can be chosen to be stable and minimum phase. The objective in this problem is to design a controller  $K$  which stabilizes  $P_0$  and minimizes the worst-case effect of any disturbance  $d$  of unit energy on the output  $y$ .

By straightforward calculation it follows that

$$y = (I + P_0 K)^{-1} W_1 d \quad (3.19)$$

and hence the objective of the uniformly optimal control problem is to minimize  $\|(I + P_0 K)^{-1} W_1\|_\infty$ . From (3.13a) this problem readily reduces to

$$\inf_{Q \in H_{\infty}} \|(\tilde{Y} - NQ) \tilde{D}W_1\|_{\infty} \quad (3.20)$$

and the optimal controller is obtained by substituting the minimizing  $Q$  in (3.20) into (3.8).

In the special case of LTI controllers, both the robust stabilization problem described in the previous section and the uniformly optimal control problem described above reduce to the same optimization problem

$$\inf_{Q \in H_{\infty}} \|A + BQC\|_{\infty} \quad (3.21)$$

where  $A$ ,  $B$ , and  $C$  are fixed stable proper transfer functions. To further interpret this problem consider the system shown in Figure 3.4.

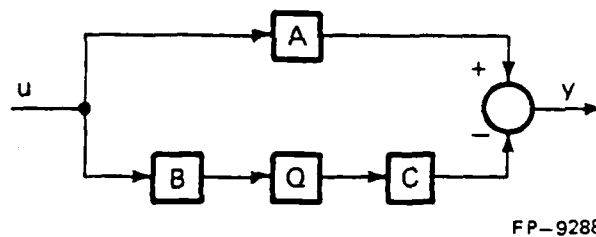


Figure 3.4. The model-matching problem.

Essentially the objective of the optimization problem (3.21) is to select  $Q$  in  $H_{\infty}$  to fit  $A+BQC$  into a disk of smallest radius in  $H_{\infty}$ . Hence, (3.21) is sometimes referred to as a **one-disk** problem. Alternatively, (3.21) may be considered as a problem of choosing the stable, proper transfer function  $Q$  such that the transfer function  $-BQC$  provides the best  $L_{\infty}$ -norm approximate of the transfer function  $A$ . Hence, this problem is also referred to as a **model-matching** problem. Any solution to (3.21) possesses the following "all-pass"



property (see Zames and Francis [1983]).

**Theorem (3.6).** Let  $Q^*$  be a solution to the one-disk problem

$$\min_{Q \in RH_\infty} \|A + BQC\|_\infty$$

where  $A$ ,  $B$ , and  $C$  are  $RH_\infty$ -functions. Let  $\gamma = \|A + BQ^*C\|_\infty$ . Then for any frequency  $\omega \in \mathbb{R}$ ,  $\|(A + BQ^*C)(j\omega)\| = \gamma$ .

The one-disk (model-matching) problem can easily be converted into the well-known **General Distance Problem** as follows (see Francis [1987]). Consider again the optimization problem (3.21) and let  $B = B_i B_o$  and  $C = C_{co} C_{ci}$  be inner-outer and co-outer-co-inner factorizations of  $B$  and  $C$ , respectively. Define the operators  $E_1$  and  $E_2$  as

$$E_1 = \begin{pmatrix} B_i^* \\ I - B_i B_i^* \end{pmatrix} \quad \text{and} \quad E_2 = (C_{ci}^* \quad I - C_{ci}^* C_{ci}). \quad (3.22)$$

It is easy to show that  $E_1$  is inner and  $E_2$  is co-inner. These operators are called **dilations** (see Doyle [1984]) and possess special norm-preserving properties. Using (3.22) the optimization problem (3.21) can be rewritten as

$$\begin{aligned} \inf_{Q \in RH_\infty} \|A + BQC\| &= \inf_{Q \in RH_\infty} \|E_1 (A + B_i B_o Q C_{co} C_{ci}) E_2\| \\ &= \inf_{\tilde{Q} \in RH_\infty} \left\| \begin{pmatrix} T_{11} - \tilde{Q} & T_{21} \\ T_{12} & T_{22} \end{pmatrix} \right\| \end{aligned} \quad (3.23)$$

where  $T_{11} = B_i^* A C_{ci}^*$ ,  $T_{12} = (I - B_i B_i^*) A C_{ci}^*$ ,  $T_{21} = B_i^* A (I - C_{ci}^* C_{ci})$ ,  $T_{22} = (I - B_i B_i^*) A (I - C_{ci}^* C_{ci})$  and  $\tilde{Q} = B_o Q C_{co}$ . This problem is known as the **4-block problem**.

The solution to the 4-block problem is iterative in nature and is commonly referred to as  $\gamma$ -iteration (see for example Chu [1985], Chu and Doyle [1985], and Francis [1987]).

In certain cases the dilation matrices  $E_1$  and  $E_2$  introduced in (3.22) are trivial. If  $B_i$  is a square full-rank matrix then  $B_i^* B_i = B_i B_i^* = I$  and hence  $E_1 = (B_i \ 0)^*$ . Similarly, if  $C_{ci}$  is square and full rank, then  $C_{ci} C_{ci}^* = C_{ci}^* C_{ci} = I$  and hence  $E_2 = (C_{ci}^* \ 0)$ . Whenever one of the operators  $B_i$  and  $C_{ci}$  are square and full rank then the 4-block problem of (3.23) may be reduced to a 2-block problem of the form

$$\inf_{Q \in RH_\infty} \left\| \begin{bmatrix} T_{11} - Q \\ T_{12} \end{bmatrix} \right\|_\infty \quad (3.24)$$

or

$$\inf_{Q \in RH_\infty} \|T_{11} - Q \ T_{21}\|_\infty.$$

These problems are also solved via the methods of  $\gamma$ -iteration.

If, however, both of the operators  $B_i$  and  $C_{ci}$  are square full-rank, then the 4-block problem of (3.23) reduces to a basic  $L_\infty/H_\infty$  approximation problem of the form

$$\inf_{H \in RH_\infty} \|G - H\|_\infty \quad (3.25)$$

where  $G = B_i^* A C_{ci}^*$  and  $H = B_0 Q C_{co}$ . This problem can be interpreted as one of finding a purely stable transfer function matrix  $H$  which best approximates (in  $L_\infty$ -norm) the unstable transfer function matrix  $G$ . It should be noted that for the problem of uniformly (or  $H_\infty$ -) optimal control of "short-fat plants," i.e., plants with more inputs than outputs, or for the problem of robust stabilization of families of plants featuring additive or multiplicative representations of dynamic uncertainty, the matrices  $B_i$  and  $C_{ci}$  will always be square.

The basic  $L_\infty/H_\infty$  approximation problem stated in (3.25) is deeply connected to interpolation theory and the classical moment problem and has been studied by several

researchers (see for example Krein and Nudelman [1977], Nehari [1957], Nevanlinna [1919], and Sarason [1967]). A particularly elegant result due to Adamjan et al. [1978] is the following:

**Theorem (3.7).** Consider the optimization problem of finding a closest  $H_\infty$ -matrix  $H$  to a given  $L_\infty$ -matrix  $G$  as stated in (3.25). There exists a solution to this problem and this solution is characterized by

$$\|G-H\|_\infty = \|\Gamma(G)\|. \quad (3.26)$$

□

**Remark (3.5).** In general the optimal  $H$  which satisfies (3.26) is nonunique. In basic terms Theorem (3.7) states that the distance between a given noncausal system  $G$  and the nearest causal system  $H$  is given by the norm of the Hankel operator  $\Gamma(G)$  associated with  $G$ . This theorem, however, does not provide any insight into determining the optimal  $H_\infty$ -matrix  $H$ . Methods for finding the optimal  $H_\infty$ -matrix  $H$  have been developed by Kimura [1984], through classical Nevanlinna-Pick interpolation theory, and by Glover [1984] via the theory of Hankel operators.

□

## CHAPTER 4

## NONLINEAR TIME-VARYING CONTROLLERS FOR ROBUST STABILIZATION

This chapter is concerned with the problem of robust stabilization of a family of plants exhibiting dynamic uncertainty. Kimura [1984] has obtained a complete solution to this problem in the case of unstructured additive uncertainty, in the context of linear time-invariant feedback. The intent of this chapter is to investigate the relative merits of nonlinear time-varying controllers for robustly stabilizing a family of plants of the type treated by Kimura [1984]. It has been demonstrated in Khargonekar et al. [1985] that for certain robust stabilization problems, such as gain margin problems, nonlinear time-varying controllers are far superior to linear time-invariant ones. This result suggests that NLTV control may offer advantages in other robust stabilization problems as well. However, it is demonstrated in this chapter that for the problem of robust stabilization of families of plants with dynamic uncertainty, this is not the case. Intermediate to proving the above theorem are several results themselves of interest. In particular, it is shown that the small-gain theorem is both necessary and sufficient for a certain class of nonlinear operators.

The results of this chapter are presented for discrete-time systems. However, all of the results very easily extend to continuous-time systems.

The remainder of this chapter is organized as follows. Section 4.1 describes the family of plant models addressed in this chapter and subsequently details some key robust stability theorems. The main results of this chapter are contained in Section 4.2, and unstructured multiplicative uncertainty versions of these results are presented in Section 4.3.

#### 4.1. Preliminaries

As described in Chapter 2, plant modeling uncertainty may be treated by representing the true physical plant by a family  $F_\delta$  of possible plant models. An approach (although possibly conservative) to this problem is to allow the family  $F_\delta$  to feature dynamic (unstructured) uncertainty. Two common paradigms for dynamic uncertainty in plant models are the additive and multiplicative representations (see Doyle and Stein [1981]). For instance, additive unstructured uncertainty is represented by the family of plant models (2.2), while multiplicative unstructured uncertainty is represented by the family of plant models

$$F_\delta = \{(I + \Delta(z) W_2(z)) P_0(z) \mid \|\Delta(z)\| < \delta\} \quad (4.1)$$

along with identical assumptions as in (2.2).

Kimura [1984] has studied the problem of robustly stabilizing the family of plants  $F_\delta$  described by (2.2). He has obtained necessary and sufficient conditions for robust stabilizability of a family of SISO plants in the context of LTI controllers. (For the MIMO case, see Vidyasagar and Kimura [1986].) This chapter focuses on an additive unstructured family of plants [somewhat different from (2.2)] of the form

$$F_\delta = \{P_0(z) + \Delta W(z) : \|\Delta\| < \delta\} \quad (4.2)$$

where  $P_0(z)$  and  $W(z)$  are as in (2.2) but  $\Delta$  is allowed to be any NLTV, causal, stable operator with  $\|\Delta\| < \delta$ .

**Remark (4.1).** The essential difference between the family of plant models (4.2) and that of Kimura [1984] is that the family  $F_\delta$  in (4.2) allows for nonlinear perturbations  $\Delta$ , which although more conservative, may be a reasonable hypothesis for modeling errors.

(For further discussion on families of plants featuring nonlinear perturbations see Khargonekar et al. [1987] and Poolla and Ting [1987].) Also, whereas Kimura permits unstable perturbations (as long as the number of unstable poles remains unaltered), the perturbations in the family of plant models (4.2) are restricted to be **stable**. Thus, strictly speaking, the families (2.2) and (4.2) cannot be compared. This is further discussed in Poolla and Ting [1987]. It shall be assumed that the entire family of plant models (4.2) is strictly proper. This assumption is merely to ensure that any proper controller one designs results in a **well-posed** feedback system.

□

The first step in treating robust stabilization problems of families of plants  $F_\delta$  as in (4.2) is to obtain robust stabilization theorems (such as Theorem (3.3)) for families of plants featuring **nonlinear** perturbations. The following result is a straightforward generalization of the results of Chen and Desoer [1982] to the family of plant models (4.2).

**Theorem (4.1).** Consider the family of plants  $F_\delta$  defined by (4.2). Introduce stable coprime factorizations of  $P_0$  as in (3.7). Then, there exists a (fixed) LTI controller  $K$  that robustly stabilizes  $F_\delta$  if and only if

$$\inf_{Q_1: \text{LTI, causal stable}} \|W_2 \tilde{X} \tilde{D} + W_2 D Q_1 \tilde{D}\| \leq \frac{1}{\delta}. \quad (4.3)$$

**Proof.** The necessity follows directly from Chen and Desoer [1982] by noting that LTI perturbations are, in particular, NLTV. To prove the sufficiency, however, a little care must be exercised in dealing with the distributivity of nonlinear operators: suppose equation (4.3) holds. It then follows from Adamjan et al. [1978] that there exists some

LTI, causal, stable operator  $Q_2$  which achieves the infimum, i.e.,

$\|W_2\tilde{X}\tilde{D} + W_2DQ_2\tilde{D}\| \leq \frac{1}{\delta}$ . Define an LTI controller  $K = (\tilde{X} + DQ_2)(\tilde{Y} - NQ_2)^{-1}$  and let

$T = K(I + P_0K)^{-1}$ . From Theorem (3.1) it is clear that  $K$  stabilizes  $P_0$  and using (3.13b) it

follows that  $\|W_2T\| \leq \frac{1}{\delta}$ . From Figure 3.1 it is easy to see that  $e_1 = u_1 - K(u_2 + Pe_1)$ .

Since  $K$  and  $(I + KP_0)^{-1}$  are linear, this becomes

$$\begin{aligned} e_1 &= (I + KP)^{-1} (u_1 - Ku_2) = (I + T\Delta W_2)^{-1} (I + KP_0)^{-1} (u_1 - Ku_2) \\ &= W_2^{-1} (I + W_2T\Delta)^{-1} W_2 \{ (I + KP_0)^{-1} u_1 - (I + KP_0)^{-1} Ku_2 \}. \end{aligned}$$

Notice now that from the small-gain theorem (3.2),  $(I + W_2T\Delta)^{-1}$  is well defined and

stable for all  $\Delta$  with  $\|\Delta\| < \delta$ . Also, since  $W_2, W_2^{-1}, (I + KP_0)^{-1}$  and  $(I + KP_0)^{-1}K$  are all

stable, one may conclude that the operator  $[\phi_1(P,K), \phi_2(P,K)]: [u_1, u_2]^T \rightarrow e_1$  is stable.

Again, from Figure 3.1 since  $P_0$  is linear it follows that

$$\begin{aligned} e_2 &= u_2 + Pe_1 = u_2 + P_0e_1 + \Delta W_2e_1 \\ &= u_2 + P_0u_1 - P_0Ke_2 + \Delta W_2e_1 \\ &= (I + P_0K)^{-1} (u_2 + P_0u_1 + \Delta W_2e_1). \end{aligned}$$

It is now evident that since  $K$  stabilizes  $P_0$ , the operator  $[\phi_3(P,K), \phi_4(P,K)]: [u_1, u_2]^T \rightarrow e_2$  is also stable. Thus,  $\phi(P,K)$  is stable for all  $P$  in  $F_\delta$  (see Definition (3.2)), and this concludes the proof.

□

**Remark (4.2).** For NLTV controllers, the proof given above breaks down. The sufficiency argument relies only on the small-gain theorem and remains valid even for NLTV controllers. However, the necessity argument from Chen and Desoer [1982] uses

frequency domain techniques and is no longer valid. Indeed, a fundamental result of this chapter (see Theorem (4.4)) demonstrates that the small-gain theorem is in a precise sense both necessary and sufficient which enables one to prove Theorem (4.1) for NLTV controllers as well.

□

## 4.2. Main Results

In this section it is shown that if there exists a NLTV controller that robustly stabilizes the family  $F_\delta$  [see (4.2)], then there must exist a linear time-invariant controller that robustly stabilizes  $F_\delta$ . A similar result for families of plants featuring multiplicative uncertainty is shown in the next section.

For purposes of clarity, the following discussion begins by outlining the sequence of ideas and the key difficulty in the proof of this result. To this end, let  $\tilde{K}$  be some NLTV controller that robustly stabilizes the family (4.2). Suppose there does not exist any LTI robustly stabilizing controller. Then, from Theorem (4.1)

$$\inf_{K: \text{LTI, causal}} \|W_2 K (I + P_0 K)^{-1}\| > \frac{1}{\delta}.$$

It is then shown (Theorem (4.2)) that

$$\inf_{K: \text{NLTV, causal}} \|W_2 K (I + P_0 K)^{-1}\| = \inf_{K: \text{LTI, causal}} \|W_2 K (I + P_0 K)^{-1}\|.$$

Thus,  $\|W_2 \tilde{K} (I + P_0 \tilde{K})^{-1}\| > \frac{1}{\delta} + \epsilon$ . Now by applying the necessity of the small-gain theorem (Theorem (4.4)) it is possible to conclude that there exists a causal, stable perturbation  $\Delta$  with  $\|\Delta\| < \delta$  and such that  $(I + \Delta W_2 \tilde{K} (I + P_0 \tilde{K})^{-1})^{-1}$  is unstable. This in



turn implies that  $\hat{K}$  does not robustly stabilize the family  $F_\delta$ .

Central to proving this main result is the necessity of the small-gain theorem (Theorem (4.4)). Given a (nontrivial) NLTV operator  $M$  with  $\|M\| > \frac{1}{\delta} + \epsilon$  it is quite easy to construct a noncausal operator  $\Delta$  with  $\|\Delta\| < \delta$  and such that  $(I + \Delta M)^{-1}$  is unstable. The essential difficulty lies in exhibiting a causal destabilizing perturbation  $\Delta$ .

In order to prove the main result, Theorem (4.5), it is necessary to derive several intermediate results, some of which are themselves of interest. To begin one may establish the following:

Theorem (4.2). Let  $A$ ,  $B$ , and  $C$  be causal, stable, linear time-invariant operators with  $C$  being square and full rank. Let  $B = B_1 B_0$  and  $C = C_0 C_1$  be inner-outer and outer-inner factorizations of  $B$  and  $C$ , respectively. Then,

$$\inf_{Q_1: \text{NLTV, causal, stable}} \|A + BQ_1C\|_{\text{bh}_2} = \inf_{Q_2: \text{LTI, causal, stable}} \|A + BQ_2C\| = \|B_1^* A C_1^*\|_H. \quad (4.4)$$

**Proof.** The proof essentially follows from arguments due to Khargonekar [1985]. First note that since  $B_1$  and  $C_1$  are inner (see Arveson [1975]),  $B_1^* B_1 = I$  and  $C_1^* C_1 = I$ . However, except in the trivial case,  $C_1 C_1^* \neq I$  because these operators are defined on right-sided sequences. Let  $N$  be some (fixed) integer and define the operator  $C_N = C_1^* z^{-N}$ . Notice that  $C_N$  is a contraction and define the operator  $\phi_N$  by

$$C_1 C_N = z^{-N} I + \phi_N. \quad (4.5)$$

It is easy to see that  $\lim_{N \rightarrow \infty} \|\phi_N\| = 0$ . Let  $Q_1$  be any NLTV, causal, stable operator, and

notice that since  $B_1^*$  and  $C_N$  are contractions,

$$\begin{aligned}\|A + BQ_1C\| &= \|A + B_1B_0Q_1C_0C_1\| \\ &\geq \|B_1^*AC_N + B_0Q_1C_0C_1C_N\|.\end{aligned}$$

Let  $Q_2 := B_0Q_1C_0$  (which is a causal, stable NLTV operator). Then, from (4.5),

$$\begin{aligned}\|A + BQ_1C\| &\geq \|B_1^*AC_N + Q_2(z^{-N}I + \phi_N)\| \\ &\geq \|P_NB_1^*AC_N + P_NQ_2(z^{-N}I + \phi_N)\| \\ &= \|P_NB_1^*AC_N + P_NQ_2\phi_N\|,\end{aligned}$$

where the last equality is a consequence of the causality of  $Q_2$ . Continuing, it follows that

$$\begin{aligned}\|A + BQ_1C\| &\geq \|P_NB_1^*AC_1^*z^{-N}\| - \|P_NQ_2\phi_N\| \\ &\geq \|P_NB_1^*AC_1^*z^{-N}\| - \|\phi_N\|K,\end{aligned}\tag{4.6}$$

where  $K$  is some constant independent of  $N$ . It is easy to see that  $\|P_NB_1^*AC_1^*z^{-N}\| = \|P_NB_1^*AC_1^*(I - P_N)\|$ . Taking limits as  $N \rightarrow \infty$  and noticing that  $\lim_{N \rightarrow \infty} \|\phi_N\| = 0$ , one immediately obtains

$$\|A + BQ_1C\| \geq \lim_{N \rightarrow \infty} \|P_NB_1^*AC_1^*(I - P_N)\| = \|B_1^*AC_1^*\|_H.$$

Since  $B_1^*$ ,  $A$ ,  $C_1^*$ , and  $P_N$  are linear, it follows that there exists an input sequence  $u$  of arbitrary norm such that  $\|(A + BQ_1C)u\| / \|u\| \geq \|B_1^*AC_1^*\|_H$ . Thus,

$$\inf_{Q_1: \text{NLTV, causal, stable}} \|A + BQ_1C\|_{\text{bh}_2} \geq \|B_1^*AC_1^*\|_H.\tag{4.7}$$

In point of fact, it is shown in Adamjan et al. [1978] that

$$\inf_{Q_1: \text{LTI, causal, stable}} \|A + BQ_1C\| = \|B_1^*AC_1^*\|_H.$$

Observing that LTI operators are, in particular, NLTV enables one to conclude equality in (4.7), proving the proposition. □

**Remark (4.3).** It is easy to see from equation (4.10) that given any  $\epsilon > 0$  and any real number  $K > 0$ , there exists a fixed input sequence  $u$  in  $h_2$  such that

$$\frac{\|(A + BQC)u\|}{\|u\|} \geq \|B_i^* A C_i^*\|_H - \epsilon,$$

for any NLTV causal operator  $Q$  with  $\|Q\| < K$ . It should be noted that a result similar to Theorem (4.2) is key in demonstrating that as far as problems of uniformly (or  $H^\infty$ -) optimal control for LTI plants are concerned, NLTV controllers offer no advantage over LTI controllers (see also Khargonekar and Poolla [1986]). □

The next step is to show that operators of the form  $M = A + BQC$  achieve their norm on some input sequence  $u$  of unbounded energy. This is demonstrated in the following key result.

**Theorem (4.3).** Let  $M$  be any  $m$ -input,  $p$ -output NLTV operator of the form  $M = A + BQC$  where  $A$ ,  $B$ , and  $C$  are causal, stable linear time-invariant operators with  $C$  being square, full rank and  $Q$  is any causal, incrementally stable NLTV operator. Let  $B = B_i B_0$  and  $C = C_0 C_i$  be inner-outer and outer-inner factorizations of  $B$  and  $C$ , respectively. Suppose further that

$$\|B_i^* A C_i^*\|_H > \frac{1}{\delta} + \epsilon. \quad (4.8)$$

Then there exists an input sequence  $u$  in  $h_{2\epsilon}^m$  such that

$$\lim_{N \rightarrow \infty} \|P_N u\| = \infty, \text{ and} \quad (4.9)$$

$$\lim_{N \rightarrow \infty} \frac{\|P_N M u\|}{\|P_N u\|} > \frac{1}{\delta} + \epsilon. \quad (4.10)$$

Proof. See Appendix. □

**Remark (4.4).** Essentially, the above theorem contains a "maximum modulus principle" for a certain class of **nonlinear time-varying** operators, i.e., it states that the norm of  $M$  is achieved on the "boundary" of  $h_2$  (i.e.,  $h_{2c}$ ). Recall that for **LTI** stable operators, the maximum modulus principle asserts that the norm of the operator is achieved on some unbounded sinusoidal input (i.e., the  $j\omega$  axis). In point of fact it can be shown that the input sequence  $u$  constructed in Theorem (4.3) can be guaranteed to be **periodic**. Notice also that condition (4.8) implies that  $\|M\| > \frac{1}{\delta} + \epsilon$  by Theorem (4.2). □

The above result is used to prove one of the key results of this section. In particular, it is now shown that the small-gain theorem is **necessary and sufficient** for nonlinear time-varying operators of a certain form.

**Theorem (4.4).** Let  $M$  be any NLTV operator with  $\|M\| > \frac{1}{\delta}$  as in Theorem (4.3).

Then, there exists a NLTV, strictly causal, stable operator  $\Delta$  with  $\|\Delta\| < \delta$ , and such that the operator  $(I + \Delta M)^{-1}$  is well-defined and unstable.

**Proof.** It follows from Theorem (4.3) that there exists an input sequence  $u$  in  $h_{2c}^m$  such that equations (4.9) and (4.10) hold. This in turn implies that there exists an integer  $N_0$  such that

$$\frac{\|P_N M u\|}{\|P_N u\|} > \frac{1}{\delta} + \epsilon, \quad N \geq N_0. \quad (4.11)$$

Let  $Mu = (y_0, y_1, y_2, \dots) = y$  and define the sequence  $\hat{u} = -(0, 0, 0, \dots, 0, u_{N_0}, u_{N_0+1}, \dots)$ . For a sequence  $w$  in  $h_{2\epsilon}^p$ , define the integers  $N(w) =$  smallest integer  $N$  such that  $w_N \neq y_N$ ;  $I(w) =$  smallest integer  $i$  such that  $w_i \neq 0$ . Define the operator  $\Delta : h_{2\epsilon}^p \rightarrow h_{2\epsilon}^p$  by

$$\Delta(w) = \begin{cases} P_{N(w)}(\hat{u}) & \text{if } N(w) \neq 0 \\ z^{-I(w)} \Delta(z^{I(w)} w) & \text{if } I(w) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.12)$$

It is easy to verify that (4.12) defines a strictly proper, stable, nonlinear time-invariant operator  $\Delta$ , and, it follows from (4.12), that  $\|\Delta\| \leq \frac{1}{\delta + \epsilon} < 1$ . Also,  $(I + \Delta M)^{-1}$  is well-defined (since  $\Delta$  is strictly proper) and

$$(I + \Delta M)u = u + \Delta y = u + \hat{u} = (u_0, u_1, \dots, u_{N_0-1}, 0, \dots).$$

Thus,  $(I + \Delta M)^{-1}(u_0, u_1, \dots, u_{N_0-1}, 0, 0, \dots) = u$ , and since  $u$  is an unbounded sequence (see equation (4.9)), it follows that  $(I + \Delta M)^{-1}$  is an unstable operator. This completes the proof. □

**Remark (4.5).** This proposition demonstrates the necessity of the small-gain theorem for a certain class of NLTV operators (see also Vidyasagar [1978]). The construction of the operator  $\Delta$  above basically involves determining a causal inverse of  $M$  on the input  $u$ , which exists because of (4.11). Essentially this construction of  $\Delta$  involves "inverting"  $M$  causally on this input  $u$ . To do this a very special choice of the input  $u$  must be selected in order to ensure that  $\|\Delta\| < \delta$ . As pointed out earlier, determining a noncausal

destabilizing perturbation  $\Delta$  is quite easy, and the essential problem is in ensuring that  $\Delta$  is causal. Ideally, one would like to determine a linear time-invariant destabilizing perturbation  $\Delta$ . This problem appears at first glance to be formidable (see also Chapter 7).

□

The central result of this section can now be stated.

**Theorem (4.5).** Consider an unstructured additive family of plants  $F_\delta$  described by (4.2). Suppose there exists a NLTV controller  $K$  that incrementally stabilizes the nominal plant  $P_0$  and robustly stabilizes  $F_\delta$ . Then, there must also exist a LTI controller  $\hat{K}$  that robustly stabilizes  $F_\delta$ .

**Proof.** (By contradiction.) Let  $K$  be some NLTV controller that robustly stabilizes  $F_\delta$  and let  $Q$  be its corresponding incrementally stable Youla parameter (see Theorem (3.1)). In particular, it follows that  $K(I - P(-K))^{-1}$  is stable for all  $P$  in  $F_\delta$  (see Remark (3.1)). Thus, since  $\Delta$  and  $W$  are also stable,

$$\psi = I + \Delta W(-K)(I - (P_0 + \Delta W)(-K))^{-1}$$

is stable for all  $\Delta$  with  $\|\Delta\| < 1$ . Notice now that

$$\begin{aligned} \psi &= I + \Delta W(-K)(I + P_0 K)^{-1} (I - \Delta W(-K)(I + P_0 K)^{-1})^{-1} \\ &= (I - \Delta W(-K)(I + P_0 K)^{-1})^{-1}. \end{aligned} \quad (4.13)$$

Also, from (3.13b),  $K(I + P_0 K)^{-1} = \tilde{X}\tilde{D} + DQ\tilde{D}$ , where  $D$ ,  $\tilde{X}$ , and  $\tilde{D}$  are obtained from stable coprime factorizations of the nominal plant  $P_0$ . With  $A = -W\tilde{X}\tilde{D}$ ,  $B = -WD$ ,  $C = \tilde{D}$  it now follows from (4.13) that

$$(I - \Delta M)^{-1} \text{ is stable for all } \Delta \text{ with } \|\Delta\| < \delta, \quad (4.14)$$

where  $M = A + BQC$ .

Now suppose that there does not exist any linear time-invariant controller that robustly stabilizes  $F$ . This implies, from Theorem (3.3), that

$$\inf_{Q_1: \text{LTI, causal stable}} \|A + BQ_1C\| > \frac{1}{\delta} + \epsilon. \quad (4.15)$$

However, from Adamjan et al. [1978] it follows that

$$\inf_{Q_1: \text{LTI, causal, stable}} \|A + BQ_1C\| = \|B_1^* A C_1^*\|_H.$$

Therefore,  $\|B_1^* A C_1^*\|_H > \frac{1}{\delta} + \epsilon$ , and consequently from Theorem (4.4), one may conclude that there exists a causal, stable operator  $\Delta$  with  $\|\Delta\| < \delta$  and such that  $(I - \Delta M)^{-1}$  is unstable. This renders (4.14) impossible, concluding the proof.

□

**Remark (4.6).** The above theorem demonstrates that nonlinear time-varying controllers provide no advantage over linear time-invariant controllers for the problem of robustly stabilizing a family of plants featuring dynamic uncertainty (see (4.2)). In general, adaptive controllers are nonlinear and time-varying; therefore, this result implies that adaptive controllers are not useful for robustly stabilizing a family of plants featuring dynamic uncertainty. Due to the proliferation of controller synthesis methods, results of this type are invaluable in ruling out various synthesis methods for certain classes of control problems.

The results of Theorem (4.5) are not entirely unexpected for the following intuitive reason. Due to the nature of dynamic uncertainty (i.e., possibly unbounded McMillan degree of the true plant model) it is impossible to further identify the true plant via input-output experiments. Hence, a nonlinear time-varying (or adaptive) control strategy will not prove advantageous.

Remark (4.7). The above theorem makes critical use of the technical restriction that the NLTV robustly stabilizing controller **incrementally** stabilizes the nominal plant. This hypothesis translates to stating that the closed-loop system composed of the NLTV controller and the **nominal plant** be "smooth," which is a reasonable requirement. An interesting open question to examine is whether or not Theorem (4.5) holds for arbitrarily NLTV controllers.

### 4.3. The Case of Multiplicative Uncertainty

This section extends the results of the previous section to families of plants featuring unstructured multiplicative uncertainty. Here, the family of plant models [somewhat different from (4.1)] is of the form

$$F_\delta = \{(1 + \Delta) P_0(z) : \|\Delta\| < \delta\} \quad (4.16)$$

where  $P_0(z)$  and  $\Delta$  are as in (4.2).

For the family of plants (4.1) featuring multiplicative unstructured uncertainty, one may derive the following robust stability criteria analogous to the results of Theorem (4.1).

Theorem (4.6). Consider the family of plants  $F_\delta$  defined by (4.1). Introduce stable coprime factorizations of  $P_0$  as in (3.7). Then there exists a (fixed) LTI controller  $K$  that robustly stabilizes  $F_\delta$  if and only if

$$\inf_{Q_1: \text{LTI, causal, stable}} \|NX + NQ_1\hat{D}\| \leq \frac{1}{\delta}.$$

The proof of this result is very similar to that of Theorem (4.1) and is hence omitted. Again, however, the proof breaks down in the case of NLTV controllers (see Remark



(4.2)), and it is necessary to employ the results of Theorem (4.4) in order to generalize these results to NLTV controllers. The following necessary condition for the robust stabilizability of a family of plants  $F_\delta$  as described by (4.16) can now be stated.

**Theorem (4.7).** Consider the family of plants  $F_\delta$  described by (4.16) and let the nominal plant  $P_0$  admit a stable coprime factorization as in (3.7). Let  $M = -P_0K(I+P_0K)^{-1} = -N(X+Q\tilde{D})$  be expressed linearly in  $Q$  as  $M = A+BQC$  by setting  $A = -NX$ ,  $B = -N$ , and  $C = \tilde{D}$ . Then there exists a NLTV controller  $K$  that robustly stabilizes  $F_\delta$  only if the operator  $C(I-\Delta M)^{-1}$  is stable for all  $\Delta$  with  $\|\Delta\| < \delta$ .

**Proof.** Suppose  $K$  is a NLTV operator that robustly stabilizes the family of plants  $F_\delta$ . From Remark (3.1) it follows that for any plant  $P$  in  $F_\delta$  the operators  $(I-P(-K))^{-1}$  and  $K(I-P(-K))^{-1}$  are both stable. It is straightforward to express

$$\begin{aligned}(I - P(-K))^{-1} &= (I - (I + \Delta) P_0(-K))^{-1} = (I + P_0K - \Delta P_0(-K))^{-1} \\ &= (I + P_0K)^{-1} (I - \Delta M)^{-1}\end{aligned}$$

$$K(I - P(-K))^{-1} = K(I + P_0K)^{-1} (I - \Delta M)^{-1}.$$

Employing (3.13a) and (3.13b) and noting that the addition and composition of stable operators is stable it follows that

$$\begin{aligned}\tilde{N}K(I-P(-K))^{-1} + \tilde{D}(I-P(-K))^{-1} &= [\tilde{N}(\tilde{X} + DQ) \tilde{D} + \tilde{D}(\tilde{Y} - NQ) \tilde{D}] (I - \Delta M)^{-1} \\ &= \tilde{D}(I - \Delta M)^{-1}\end{aligned}$$

is indeed a stable operator for all  $\Delta$  with  $\|\Delta\| < \delta$ . This completes the proof. □

One may now derive the following result.

**Theorem (4.8).** Consider the problem of robustly stabilizing the family of plants  $F_\delta$

described in (4.16). Suppose there exists a NLTV controller  $K$  that incrementally stabilizes the nominal plant  $P_0$  and robustly stabilizes  $F_\delta$ . Then, there must also exist a LTI controller  $\hat{K}$  that robustly stabilizes  $F_\delta$ .

**Proof.** (By Contradiction). Let the operator  $M = A + BQC$  be as defined in Theorem (4.7). Let  $K$  be some NLTV controller that robustly stabilizes  $F_\delta$  and let  $Q$  be its corresponding incrementally stable Youla parameter (see Theorem (3.1)). From Theorem (4.6) this implies that the operator  $C(I - \Delta M)^{-1}$  is stable for all  $\Delta$  with  $\|\Delta\| < \delta$ .

Now assume that there does not exist any LTI controller  $\hat{K}$  that robustly stabilizes  $F_\delta$ . This implies, from Theorem (3.3), that for some fixed  $\epsilon > 0$

$$\inf_{Q \text{ LTI, causal, stable}} \|A + BQC\| = \|B_1^* A C_1^*\|_H > \frac{1}{\delta} + \epsilon.$$

Consequently, from Theorem (4.3) one may conclude that there exists an input sequence  $u$  in  $h_{2c}^m$  satisfying (4.9) and (4.10) and such that  $Mu = y$ . Following Theorem (4.4) it is possible to define a stable causal operator  $\Delta$  similar to (4.12) such that  $\|\Delta\| < \frac{1}{\delta + \epsilon} < \frac{1}{\delta}$  and  $(I - \Delta M)^{-1}$  is well-defined and unstable.

The operator  $(I - \Delta M)^{-1}$  maps  $(I - \Delta M)^{-1}(u_0, u_1, \dots, u_{N_0-1}, 0, 0, \dots) = u$  and thus  $C(I - \Delta M)^{-1}(u_0, u_1, \dots, u_{N_0-1}, 0, 0, \dots) = Cu$ . Now suppose that the sequence  $Cu$  is bounded. It is known that

$$\begin{aligned} Mu &= (A + BQC)u = Au + BQCu = -NXu + BQCu \\ &= -(I - \tilde{Y}\tilde{D})u + BQCu = -u + \tilde{Y}\tilde{D}u + BQ\tilde{D}u. \end{aligned}$$

Thus, if  $Cu = \tilde{D}u$  is bounded then  $\tilde{Y}\tilde{D}u + BQ\tilde{D}u$  is bounded since  $\tilde{Y}$ ,  $B$ , and  $Q$  are all stable operators. Hence,

$$Mu = -u + v \quad \text{where } v \in h_2.$$

This implies that  $\|Mu\|/\|u\| = 1$  which contradicts the fact that the input sequence  $u$  was chosen such that  $\frac{\|Mu\|}{\|u\|} > \frac{1}{\delta} + \epsilon$ . Therefore, the sequence  $Cu$  cannot be bounded.

Now assume that the sequence  $Cu$  is unbounded. Then the operator  $C(I-\Delta M)^{-1}$  which maps  $C(I-\Delta M)^{-1} (u_0, u_1, \dots, u_{N_0-1}, 0, 0, \dots) = Cu$  is an unstable operator. From Theorem (4.6) it follows that there does not exist any NLTV robustly stabilizing controller for  $F_\delta$  and this concludes the proof.

□

## CHAPTER 5

### ROBUST STABILIZATION PROBLEMS WITH ADDITIVE BLOCK-STRUCTURED UNCERTAINTY

The research in this chapter is focused upon various aspects of the robust stabilization problem with additive block-structured uncertainty. As is well-known, the solution of this problem reduces to an optimization problem of the form

$$\inf_{\Lambda, \Lambda^{-1}, Q \text{ in } H_{\infty}} \|\Lambda(A + BQC)\Lambda^{-1}\|_{\infty}. \quad (5.1)$$

Here,  $\Lambda$  is constrained to be block-diagonal conformally to the uncertainty structure and  $A$ ,  $B$ , and  $C$  are determined from a stable coprime factorization of the nominal plant model. An approach to the solution of this **infinite dimensional** optimization problem is the so-called  $\Lambda$ - $Q$  iteration of Doyle [1983]. The basic idea involved here is to alternately freeze the scaling  $\Lambda$ , find the best  $Q$ , freeze  $Q$ , find the best  $\Lambda$  and so on. While the above problem is convex individually in  $\Lambda$  and in  $Q$ , it fails to be jointly convex. Hence there is no guarantee that this procedure will converge to a global minimum. Further,  $\Lambda$ - $Q$  iteration has some numerical drawbacks. For instance, at each step, the optimal scaling  $\Lambda(j\omega)$  must be computed at each frequency. Also,  $\Lambda(j\omega)$  must then be approximated rationally in order to determine the optimum  $Q$ .

In this chapter the infinite dimensional optimization problem presented above is reduced to a nonlinear optimization problem involving a finite number of variables. This reduction is accomplished by exploiting a key observation due to Safonov [1986] that the optimization problem in (5.1) is dependent only on the values of  $\Lambda$  and  $\Lambda^{-1}$  at the unstable poles of the nominal plant model  $P_0$ . Using this fact it is possible to significantly reduce the computational effort involved in this controller design problem. Also, this

approach provides insight into the robust stabilization problem with additive block-structured uncertainty. In particular, some simple sufficient conditions on the nominal plant  $P_0$  are derived which guarantee infinite additive stability margin.

In a recent paper, Safonov [1986] has treated a more general optimization problem of which robust stabilization with block-structured uncertainty is a special case. He has been able to show that this more general problem also reduces to nonlinear programming with a finite number of variables. However, when these results are specialized to the robust stabilization problem treated in this chapter, the explicit dependence of this optimization problem on the scaling function residues is complex. Thus, one of the contributions of this chapter may be viewed as a simplification of the particular nonlinear programming problem that must be solved for robust stabilization with block-structured uncertainty.

A key step in recovering the controller designed via the techniques of this chapter is the determination of a stable, minimum phase (outer) function subject to certain interpolation constraints at the unstable poles of  $P_0$ . The degree of the resulting controller is critically dependent on the order of these outer functions. A new method for computing these interpolants that yields far lower-order interpolants than the method of Youla et al. [1974] shall be presented.

The remainder of this chapter is organized as follows. Section 5.1 contains a detailed account of the residue iteration algorithm and some remarks on the numerical aspects of this problem. The case of infinite stability margin is examined in Section 5.2. Following this, in Section 5.3 these results are illustrated via a few numerical examples.

## 5.1. Main Results

It will be shown in this section that the problem of robustly stabilizing an additive structured family of square plants reduces to a computationally tractable linear algebraic optimization problem referred to as residue iteration. As pointed out in Chapter 3, many families of plant models (possibly non-square) with block-structured uncertainties can be transformed into equivalent fictitious square families of plant models with block-structured uncertainties. Some numerical aspects of this problem shall also be addressed. As stated earlier, attention shall be focused on the case of three or fewer blocks in order that the results obtained are both necessary and sufficient. Also, the following exposition shall deal only with scalar block sizes (i.e.,  $r_i=1$  for  $i=1,2,\dots,k$ ) to avoid cumbersome notation though there is no essential loss of generality by doing so. For additional information on the results presented in this chapter see Ting, Cusumano, and Poolla [1987].

### 5.1.1. The Residue Iteration Algorithm

Beginning with the alternate characterization of robust stabilizability as in Theorem (3.4) the following result is easily obtained.

**Theorem (5.1).** Consider the family of plants (2.3) with block-structured uncertainty. Then, there exists a controller  $K(z)$  that robustly stabilizes the family (2.3) if and only if

$$\sup_{\Lambda \in \Omega} \underline{\sigma}(\Gamma(\Lambda P_0 W^{-1} \Lambda^{-1})) > \delta. \quad (5.2)$$

**Proof:** From Theorem (3.5) it follows that there exists a robustly stabilizing controller if and only if inequality (3.18) holds. Define the transfer function  $P_\Lambda(s) = \Lambda P_0 \Lambda^{-1}$ . It can be verified that a stably coprime factorization of  $P_\Lambda$  is

$$P_{\Lambda} = (\Lambda N \Lambda^{-1})(\Lambda D \Lambda^{-1})^{-1} = (\Lambda \tilde{D} \Lambda^{-1})^{-1} (\Lambda \tilde{N} \Lambda^{-1})$$

$$(\Lambda \tilde{N} \Lambda^{-1})(\Lambda \tilde{X} \Lambda^{-1}) + (\Lambda \tilde{D} \Lambda^{-1})(\Lambda \tilde{Y} \Lambda^{-1}) = I.$$

It is now evident from Theorem (3.3) that inequality (3.18) holds if and only if the unstructured family of plant models  $\{P_{\Lambda} + \Delta W : \|\Delta\| \leq \delta\}$  is robustly stabilizable. Now from Theorem (3.4) this family is robustly stabilizable if and only if inequality (5.2) holds, completing the proof. □

It is crucial to note that  $\Gamma(\Lambda P_0 W^{-1} \Lambda^{-1})$  depends only on the residues of  $\Lambda$  at the unstable poles of  $P_0$ . The following key proposition demonstrates the relationship between the smallest Hankel singular value of  $\Lambda P_0 W^{-1} \Lambda^{-1}$  and the residues of  $\Lambda$  at the unstable poles of  $P_0$ .

**Proposition (5.1).** Let  $\pi(A, B, C, O)$  be a canonical realization of  $(P_0 W^{-1})_+$  with  $A$  in Jordan form and let the McMillan degree of  $(P_0 W^{-1})_+ = N$ . Let  $\Lambda(s) = \text{diag}\{d_1(s), d_2(s), \dots, d_k(s)\}$  and for  $i=1, 2, \dots, k$  define the  $N \times N$  residue matrices  $R_i$  by

$$R_i = \text{diag}\{d_i(p_1), d_i(p_2), \dots, d_i(p_N)\}$$

where the diagonal elements of  $A$  are  $p_1, p_2, \dots, p_N$ . For  $i=1, 2, \dots, k$  let  $b_i = i$ th column of  $B$  and let  $c_j = j$ th row of  $C$ , and solve the Lyapunov equations

$$A M_i + M_i A^T = b_i b_i^T$$

$$A^T W_i + W_i A = c_i^T c_i$$

(5.3)

for the grammians  $M_i$  and  $W_i$ . Define

$$M(R) = \sum_{i=1}^k R_i^{-1} M_i R_i^{-1} \quad (5.4)$$

$$W(R) = \sum_{i=1}^k R_i W_i R_i.$$

Then,

$$\underline{\sigma}[\Gamma(\Lambda P_0 W^{-1} \Lambda^{-1})] = \underline{\Lambda}^{1/2} [M(R)W(R)].$$

**Proof.** Let  $e_i = i^{\text{th}}$  column of the  $k \times k$  identity matrix  $I_k$ . With the above (tedious!) notation, it follows from elementary matrix algebra that a realization for  $(\Lambda P_0 W^{-1} \Lambda^{-1})_+$  is  $\pi(A, B_\Lambda, C_\Lambda, O)$  where

$$B_\Lambda = \sum_{i=1}^k R_i^{-1} b_i e_i^T \quad C_\Lambda = \sum_{i=1}^k e_i c_i R_i.$$

Also notice that since  $\{e_i : i = 1, 2, \dots, k\}$  is an orthonormal set,

$$B_\Lambda B_\Lambda^T = \sum_{i=1}^k R_i^{-1} b_i b_i^T R_i^{-1}, \quad C_\Lambda^T C_\Lambda = \sum_{i=1}^k R_i c_i^T c_i R_i. \quad (5.5)$$

Let  $M(R)$  and  $W(R)$  be solutions of the Lyapunov equations

$$\begin{aligned} AM(R) + M(R)A^T &= B_\Lambda B_\Lambda^T \\ A^T W(R) + W(R)A &= C_\Lambda^T C_\Lambda. \end{aligned} \quad (5.6)$$

Then from well-known results (see Glover [1986]),  $\underline{\sigma}(\Gamma(\Lambda P_0 W^{-1} \Lambda^{-1})) = \underline{\Lambda}^{1/2} (M(R)W(R))$ .

It remains to be shown that the solutions of the Lyapunov equations (5.6) are given by the equations (5.4). The demonstration of this fact rests on the key observation that for  $i = 1, 2, \dots, k$ ,  $A$  and  $R_i$  commute. To see this, let  $J_1, J_2, \dots, J_l$  be the Jordan blocks of  $A$  and partition  $R_i$  conformally with these blocks as



$$AR_i = \begin{bmatrix} J^1 & & \\ & J^2 & \\ & & \ddots \\ & & & J^1 \end{bmatrix} \begin{bmatrix} R_i^1 & & \\ & \ddots & \\ & & R_i^1 \end{bmatrix}.$$

Notice now that  $R_i^j = d_i(p_j)I_{r_j}$  where  $p_j$  is the pole corresponding to the  $r_j \times r_j$  Jordan block  $J^j$ . Thus, clearly  $AR_j = R_j A$ . Now to complete the proof observe that

$$\begin{aligned} & A^T \left( \sum_{i=1}^k R_i W_i R_i \right) + \left( \sum_{i=1}^k R_i W_i R_i \right) A \\ &= \sum_{i=1}^k R_i (A^T W_i + W_i A) R_i \\ &= \sum_{i=1}^k R_i c_i^T c_i R_i = C_A^T C_A. \end{aligned}$$

Thus, the solution  $W(R)$  to the Lyapunov equation (5.6) is given by  $W(R) = \sum_{i=1}^k R_i W_i R_i$ .

One can similarly derive the second part of equation (5.4) completing the proof. □

The above proposition suggests an algorithm for solving the robust stabilization problem with additive block-structured uncertainty which is outlined below.

Consider the family of plant models (2.3).

**Step 1.** Compute a canonical realization  $\pi(A, B, C, O)$  for  $(P_0 W^{-1})_+$  with  $A$  in Jordan form, say,

$$A = \begin{bmatrix} p_1 & * & & \\ & p_2 & * & \\ & & \ddots & * \\ & & & p_N \end{bmatrix}.$$

Step II. For  $i=1,2,\dots,k$ , let  $b_i$  =  $i$ th column of  $B$  and let  $c_i$  =  $i$ th row of  $C$  and solve the Lyapunov equations (5.3) for the grammians  $M_i$  and  $W_i$ .

Step III. Compute

$$\sup_{R_1 \cdots R_k} \underline{\lambda}^{1/2}(M(R)W(R)) = \gamma \quad (5.7)$$

where  $M(R)$  and  $W(R)$  are as in (5.4). Let  $R_1^0, \dots, R_k^0$  be a solution to this optimization problem.

Step IV. For  $i=1,2,\dots,k$  find outer (stable, minimum phase) functions  $d_i(s)$  that interpolate  $p_i$  to  $R_i^0(j,j)$ . Such interpolants always exist (see Youla et al. [1974]). Define  $\Lambda(s) = \text{diag}\{d_1(s), \dots, d_k(s)\}$ .

Step V. Let  $P_u = (\Lambda P_0 W^{-1} \Lambda^{-1})_+$  and let  $P_s = \Lambda P_0 W^{-1} \Lambda^{-1} - P_u$ . Determine via the methods of Glover [1984] an optimal  $N$ -1st order anticausal Hankel norm approximant  $P_{N-1}$  for  $P_u$  such that  $P_u - P_{N-1}$  is inner, where  $N$  = McMillan degree of  $P_u$ .

Step VI. Then an optimal robustly stabilizing controller for the family (2.3) is given by

$$K(s) = -\Lambda^{-1}(P_{N-1} + P_s)^{-1} \Lambda. \quad (5.8)$$

The above residue iteration algorithm is illustrated in Section 5.4 via a few examples. The development now shifts to address some important practical details concerning this algorithm.

### 5.1.2. Numerical Aspects of Residue Iteration

The key numerical bottleneck in the above algorithm is, of course, Step III. In general it is not true that the functional  $\underline{\lambda}^{1/2}(M(R)W(R))$  is concave in the parameter vector  $R = (R_1, R_2, \dots, R_k)$ . Trivial counterexamples to demonstrate this can be easily generated. This algorithm has been implemented using a multidimensional line search without derivatives such as the method of Hooke and Jeeves (see Bazaraa and Shetty [1979]). Several numerical examples indicate that the functional  $\underline{\lambda}^{1/2}(M(R)W(R))$  may, however, be quasiconcave in  $R$  (see Rockafellar [1970]). This would be very nice since quasiconcave functions have at most one local maximum. Thus, upon arrival at a local maximum in Step III one need only check the boundary behavior of  $\underline{\lambda}^{1/2}(M(R)W(R))$  as  $R$  approaches  $\infty$  to determine the global maximum. These issues are still under investigation.

In general, the residues  $d_i(p_j)$   $i = 1, 2, \dots, k$ ,  $j = 1, 2, \dots, N$  will be complex. For real poles  $p_j$  it suffices to carry out the maximization (5.7) over real positive residues. It should also be noted that as in Doyle [1982] one may, without loss of generality, choose  $d_1(s) = 1$ .

Step IV of the above algorithm is the next key step and this involves determining outer interpolating functions. The degree of these functions directly affects the McMillan degree of the controller (in Step VI) and, therefore, it is desirable to have low-order interpolants. The interpolation technique of Youla et al. [1974] is unsatisfactory in this regard. Also, the  $C^\star$  or logarithmic interpolation methods of Ball and Helton [1979], which provide minimum norm outer interpolants, yields transcendental functions which then have to be rationally approximated. The following exposition outlines a technique to determine outer interpolating functions that always yield (much) lower-order rational

outer functions than the methods of Youla and is computationally much less demanding than the methods of Ball and Helton.

Let  $\{a_i : i = 1, 2, \dots, L\}$  and  $\{b_i : i = 1, 2, \dots, L\}$  be two conjugate closed sets of complex numbers in  $H$  and  $C - \{0\}$ , respectively. Suppose further that if  $a_i$  is real, then  $b_i$  is in  $R_+$  for the corresponding integer  $i$ . Then it is well-known (see Youla et al. [1974]) that there exists an outer (stable, minimum phase) function  $f(s)$  such that

$$f(a_i) = b_i \quad i = 1, 2, \dots, L. \quad (5.9)$$

For any positive integer in  $m$  define the Nevanlinna-Pick matrix

$$N_m = \left[ \frac{b_i^{1/m} + \bar{b}_j^{1/m}}{a_i + \bar{a}_j} \right] \quad i, j = 1, 2, \dots, L$$

where  $\bar{a}$  denotes complex conjugate of  $a$  and  $b^{1/m}$  is the principal  $m^{\text{th}}$  root of  $b$ . It is now possible to show the following result.

**Theorem (5.2).** Consider the problem of outer function interpolation described above in (5.9). Let  $m_0$  be the smallest integer such that  $N_{m_0}$  is positive definite. Then there exists an outer rational function  $f(s)$  of order at most  $m_0 L$  that solves this interpolation problem. One such interpolant is given by

$$f(s) = \left[ \frac{1 + g\left(\frac{s-1}{s+1}\right)}{1 - g\left(\frac{s-1}{s+1}\right)} \right]^{m_0}.$$

Here  $g(z)$  is determined by

$$g(z) = \frac{(-1)^L \left( \sum_{i=1}^L \frac{\beta_i x_i}{z - \alpha_i} \right)}{\pi \left( \sum_{i=1}^L \frac{x_i}{z - \alpha_i} \right)} \quad (5.10)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_L \end{bmatrix} = \left[ \frac{1 - \beta_i \bar{\beta}_j}{1 - \alpha_i \bar{\alpha}_j} \right]^{-1} \begin{bmatrix} \frac{1}{\bar{\alpha}_1} \\ \frac{1}{\bar{\alpha}_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{\bar{\alpha}_L} \end{bmatrix} \quad \left( \sum_{i=1}^L \frac{x_i}{z - \alpha_i} \right)$$

$$\alpha_i = \frac{a_i - 1}{a_i + 1}, \quad \beta_i = \frac{b_i^{1/m_0} - 1}{b_i^{1/m_0} + 1}, \quad i = 1, 2, \dots, L.$$

Proof: First note that since

$$\lim_{m \rightarrow \infty} N_m = \left[ \frac{2}{a_i + \bar{a}_j} \right] > 0,$$

there must exist a smallest integer  $m_0$  such that  $N_{m_0} > 0$ . Defining  $\alpha_i$  and  $\beta_i$ ,  $i = 1, 2, \dots, L$  as in (5.10), it is easy to see using a  $H \rightarrow D$  conformal map that  $N_{m_0} > 0$  implies that

$$M = \left[ \frac{1 - \beta_i \bar{\beta}_j}{1 - \alpha_i \bar{\alpha}_j} \right] > 0. \quad (5.11)$$

This in turn implies via Krein and Nudelman [1977] that for some  $\epsilon > 0$  there exists an

analytic function  $g(z) : \bar{D} \rightarrow D_{1-\epsilon}$ . Here  $D_{1-\epsilon}$  is the open disk of radius  $1-\epsilon$  in  $\mathbb{C}$ . This result makes critical use of the positive definiteness of the matrix  $M$  (see (5.11)). One such rational function  $g(z)$  of order at most  $L$  may be determined using formulae (5.10). Notice now that

$$h(s) \triangleq \frac{1 + g\left(\frac{s-1}{s+1}\right)}{1 - g\left(\frac{s-1}{s+1}\right)} : H \rightarrow H_\delta$$

is an outer function interpolating  $h(a_i) = b_i^{1/m_0}$  for  $i = 1, 2, \dots, L$ . It is now evident that  $f(s) = h(s)^{m_0}$  solves the outer function interpolation problem (5.9) and is a rational function of order at most  $m_0 L$ .

**Example (5.1).** Consider the problem of finding an outer function  $f(s)$  such that  $f(1) = 1$ ,  $f(2) = 3$ ,  $f(3) = 2$ . It can be verified that the method of Youla et al. [1974] yields a 48th-order rational function. However, using the method of Theorem (5.2) one obtains

$$N_m = \begin{vmatrix} 1 & \frac{1+3^{\frac{1}{m}}}{3} & \frac{1+2^{\frac{1}{m}}}{4} \\ \frac{1+3^{\frac{1}{m}}}{3} & \frac{1}{3^{\frac{1}{m}}} & \frac{\frac{1}{3^{\frac{1}{m}}} + \frac{1}{4^{\frac{1}{m}}}}{3^{\frac{1}{m}} + 2^{\frac{1}{m}}} \\ \frac{1+2^{\frac{1}{m}}}{4} & \frac{\frac{1}{3^{\frac{1}{m}}} + \frac{1}{5^{\frac{1}{m}}}}{3^{\frac{1}{m}} + 2^{\frac{1}{m}}} & \frac{\frac{1}{2^{\frac{1}{m}}}}{2^{\frac{1}{m}}} \end{vmatrix}$$

and  $N_5 \geq 0$ . Thus, it is possible to compute a 15th-order outer rational function  $f(s)$  that meets the required interpolation constraints.

## 5.2. The Case of Infinite Stability Margin

In this subsection some sufficient conditions on the nominal plant  $P_0$  will be derived in order that it exhibit **infinite** additive block-structured stability margin. These results hold for an arbitrary number of blocks (even for  $\geq 3$  blocks). In particular, one may demonstrate the following interesting result:

**Theorem (5.3).** Let  $P_0$  be a transfer-function matrix with McMillan degree  $(P_0)_+ = N$ , and consider an additive block-structured family of plant models as in (2.3). Partition  $P_0$  conformally with its block diagonal uncertainty structure, let  $D_0 =$  the block diagonal part of  $P_0$ , and define  $Q_0 \triangleq P_0 - D_0$ . Suppose further that the following technical condition holds:

For  $i \neq i_1$ , and  $j \neq j_1$  the unstable poles of  $Q_0(i,j)$  and the unstable poles of  $Q_0(i_1, j_1)$  are distinct. If McMillan degree of  $(Q_0)_+ = N$  (5.12) then the maximal achievable stability margin is infinite.

**Proof.** Without loss of generality assume  $W(s) = 1$  in (2.3). As a consequence of the technical hypothesis (5.12), it follows that there exist outer functions  $\Lambda$  in  $\Omega$  with arbitrary residues  $d_2(p_j) \triangleq R_2(j,j)$ . In particular, for any fixed real number  $\alpha$  there exists an outer function  $\Lambda_\alpha$  in  $\Omega$  such that

$$(\Lambda_\alpha P_0 \Lambda_\alpha^{-1})_+ = (D_0 + \alpha Q_0)_+.$$

It now follows from Glover [1984] that

$$\gamma_\alpha \triangleq \underline{\sigma}(\Gamma(\Lambda_\alpha P_0 \Lambda_\alpha^{-1})) = \inf_{\substack{P_{N-1} : \text{of} \\ \text{McMillan degree } N-1}} \|\Gamma(\Lambda_\alpha P_0 \Lambda_\alpha^{-1} - P_{N-1})\|.$$

Suppose this infimum is achieved by  $P_{N-1}^0$ . Then,

$$\begin{aligned}
\gamma_\alpha &= \|\Gamma(\Lambda_\alpha P_0 \Lambda_\alpha^{-1} - P_{N-1}^0)\| = \|\Gamma(D_0 + \alpha Q_0 - P_{N-1}^0)\| \\
&\geq \|\Gamma(\alpha Q_0 - P_{N-1}^0)\| - \|\Gamma(D_0)\| \\
&\geq \inf_{\substack{P_{N-1}: \text{ of} \\ \text{McMillan degree } N-1}} \|\Gamma(\alpha Q_0 - P_{N-1})\| - \|\Gamma(D_0)\| \\
&= \alpha \underline{g}(Q_0) - \|\Gamma(D_0)\|,
\end{aligned}$$

where the last step above critically uses the hypothesis that the McMillan degree  $(Q_0)_+ = N$ . Since  $\underline{g}(\Gamma(Q_0)) \neq 0$  it follows that

$$\gamma = \sup_{\Lambda \in \Omega} \underline{g}(\Lambda P_0 \Lambda^{-1}) \geq \sup_{\alpha} \gamma_\alpha = \infty$$

or that (by Theorem (5.1)) the maximal achievable stability margin is infinite.

□

Essentially what the above theorem states is that if every "unstable mode" of  $P_0$  is present in the off-diagonal blocks of  $P_0$  then the maximal achievable stability margin is infinite. It is possible to prove much sharper forms of Theorem (5.3) (by weakening the technical hypothesis (5.12)); however, these results involve cumbersome notation and are not as clean as the above theorem (see Cusumano [1987]).

**Example (5.2).** Consider the family of plant models as in (2.3) with

$$P_0 = \begin{bmatrix} \frac{(2s-5)}{(s-2)(s-3)} & \frac{15}{s-3} \\ \frac{15}{s-2} & \frac{1}{s+1} \end{bmatrix}.$$



Here,  $Q_0 = 15 \begin{bmatrix} 0 & \frac{1}{s-3} \\ \frac{1}{s-2} & 0 \end{bmatrix}$  and McMillan degree  $(P_0)_+ = \text{McMillan degree } (Q_0)_+ = 2$  and

from Theorem (5.3) it is easy to see that  $P_0$  can be robustly stabilized against arbitrarily large additive block-structured perturbations.

### 5.3. Numerical Examples

This section contains a few numerical examples to illustrate some of the concepts of the previous sections.

**Example (5.3).** Consider the 2-input 2-output transfer function matrix

$$P_0 = \begin{bmatrix} \frac{1}{s-1} & \frac{2}{s-2} \\ \frac{2}{s-3} & \frac{1}{s-2} \end{bmatrix}.$$

Consider also the additive family of plant models with structured uncertainty (as in (2.3)):

$$\mathbf{F}_\delta = \left\{ P_0 + \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \|\Delta_i\|_\infty \leq \delta \right\}.$$

Suppose the objective is to find the largest stability margin  $\delta$  against which robust stabilization is possible using the residue iteration algorithm of Section 5.2.

**Step 1:** A canonical realization for  $(P_0)_+$  with  $A$  in Jordan form is  $\pi(A, B, C, D)$  where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & .5 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Step II: Solving the Lyapunov equations (5.3) for the grammians  $M_1$  and  $W_1$  one obtains

$$M_1 = \begin{pmatrix} .5 & 0 & .5 \\ 0 & 0 & 0 \\ .5 & 0 & .67 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & .25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$W_1 = \begin{pmatrix} .5 & .33 & 0 \\ .33 & .25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & .4 \\ 0 & .4 & .17 \end{pmatrix}.$$

Step III: Solving the optimization problem (5.7) yields

$$R_1 = I_3 \text{ and } R_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & .952 & 0 \\ 0 & 0 & .85 \end{pmatrix}.$$

Also, the supremum  $\gamma = 0.123$ .

Step IV: Using the method outlined in Section 5.2 it is easy to see that the Nevanlinna-Pick matrix  $N_m$  associated with this interpolation problem is

$$N_m = \begin{pmatrix} \frac{(.952)^{\frac{1}{m}}}{2} & \frac{(.952)^{\frac{1}{m}} + (.85)^{\frac{1}{m}}}{5} \\ \frac{(.952)^{\frac{1}{m}} + (.85)^{\frac{1}{m}}}{5} & \frac{(.85)^{\frac{1}{m}}}{3} \end{pmatrix}.$$

Further,  $N_1$  is positive definite and using the methods of Krein and Nudelman [1979] it is possible to find a second-order rational outer interpolant. One such function is

$$d(s) = \frac{.5103s^2 + .5746s + 4.065}{.8457s^2 + .3614s + 3.515}, \text{ and}$$

$$\Lambda(s) = \begin{bmatrix} 1 & 0 \\ 0 & d(s) \end{bmatrix}.$$

Step V: Using the methods of Glover [1984] an optimal second-order anti-causal Hankel norm approximant  $P_2(s)$  of  $(\Lambda P_0 \Lambda^{-1})_+$  may be found to be

$$P_2(s) = \frac{\begin{bmatrix} -.1149(s^2 + 21.85s + 552.7) & -.043(s^2 - 436s + 4270) \\ -.043(s^2 - 257.9s + 1042) & -.1149(s^2 - 24.7s + 658.8) \end{bmatrix}}{(s - 80.4)(s - 1.31)}.$$

Step VI: A realization for an optimal robustly stabilizing controller can be found via (5.8) to be  $K(s) = \pi(A, B, C, D)$  where

$$A = \begin{vmatrix} 0.000e+00 & 1.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 \\ -7.966e+00 & -1.126e+00 & 2.902e+01 & -5.322e+00 & -1.334e-01 & -1.093e+00 & -8.323e+00 & 1.922e+02 & -2.902e+01 & -5.322e+00 \\ 0.000e+00 & 0.000e+00 & -2.963e+00 & -4.566e-01 & -1.491e-01 & -1.222e+00 & 7.521e+00 & 8.466e-01 & 2.963e+00 & 5.434e-01 \\ 0.000e+00 & 0.000e+00 & 1.100e+01 & -1.683e+00 & 3.894e-01 & 3.190e+00 & -2.198e+01 & -1.606e+01 & -6.848e+00 & -1.256e+00 \\ 0.000e+00 & 0.000e+00 & 2.458e+01 & -4.508e+00 & 2.277e+00 & -7.195e+00 & -7.050e+00 & 1.628e+02 & -2.458e+01 & -4.508e+00 \\ 0.000e+00 & 0.000e+00 & 1.625e+01 & -2.980e+00 & 2.536e+00 & -4.128e+00 & -4.661e+00 & 1.076e+02 & -1.625e+01 & -2.980e+00 \\ 0.000e+00 & 0.000e+00 & -6.550e+00 & 1.201e+00 & -1.972e-01 & -1.615e+00 & 1.258e+01 & -9.957e+00 & 6.550e+00 & 1.201e+00 \\ 0.000e+00 & 0.000e+00 & -8.961e+00 & 1.643e+00 & 9.358e-01 & 7.668e+00 & -3.288e+01 & -9.186e+01 & 8.961e+00 & 1.643e+00 \\ 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 1.000e+00 \\ 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & 0.000e+00 & -4.156e+00 & -4.273e-01 \end{vmatrix}$$

$$B = \begin{vmatrix} 0.000e+00 & 0.000e+00 \\ -1.995e+01 & 3.166e+01 \\ -2.983e+00 & -3.233e+00 \\ 1.035e+01 & 7.472e+00 \\ -1.689e+01 & 2.682e+01 \\ -1.117e+01 & 1.773e+01 \\ -2.508e+00 & -7.146e+00 \\ 3.375e+01 & -9.776e+00 \\ 0.000e+00 & 0.000e+00 \\ 0.000e+00 & -4.156e+00 \end{vmatrix}$$

$$C = \begin{vmatrix} 0.000e+00 & 0.000e+00 & 1.601e+00 & -2.937e-01 & 2.753e-01 & 2.256e+00 & -1.205e+01 & -2.495e+01 & -1.601e+00 & -2.937e-01 \\ -9.166e-01 & -1.681e-01 & 6.982e+00 & -1.281e+00 & -3.210e-02 & -2.630e-01 & -2.002e+00 & 4.625e+01 & -6.982e+00 & -1.281e+00 \end{vmatrix}$$

$$D = \begin{vmatrix} 7.618e+00 & 1.747e+00 \\ -4.799e+00 & 7.618e+00 \end{vmatrix}$$

Example (5.4). With  $P_0$  in Example (5.3) consider the structured family of plant models

$$\mathcal{P}(\delta_1, \delta_2) = \left\{ P_0 + \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \|\Delta_i\| \leq \delta_i \right\}.$$

This example shall demonstrate the tradeoffs between the achievable stability margins  $\delta_1$  and  $\delta_2$  in the two channels. This can be done by a rescaling of the nominal plant by the ratio  $q = \frac{\delta_1}{\delta_2}$  and then using the methods of Section 5.3. Since the details involved are identical to those in the previous example only the final results shall be discussed. Plotted in Figure 5.1 is a curve  $\alpha$  (shown in the solid line) which represents the maximum achievable stability margins on each channel as a function of  $q$ .

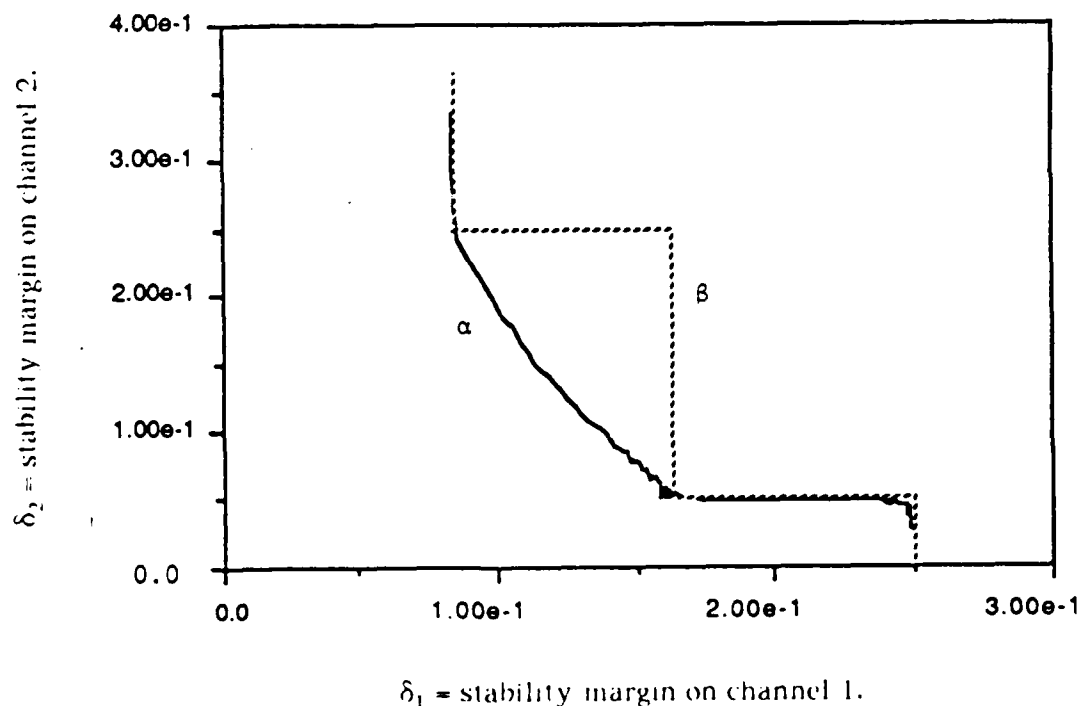


Figure 5.1. Maximum achievable stability margins on each channel.

Any desired stability margin below this curve can be obtained by some FDLTI controller  $K(s)$ . Following the works of Khargonekar et al. [1987] and Poolla and Ting [1987], an interesting open issue is the possible advantage of the use of nonlinear time-varying control towards obtaining better stability margins. Suppose one finds a perturbation pair  $(\Delta_1^0, \Delta_2^0)$  with norms  $(\|\Delta_1^0\|, \|\Delta_2^0\|) = (\delta_1^0, \delta_2^0)$  such that  $P_0 + \text{diag}\{\Delta_1, \Delta_2\}$  has one less unstable pole than  $P_0$ . Then, it follows from an elementary application of the arguments of Khargonekar et al. [1987] that there cannot exist any (possibly) nonlinear time-varying controller that yields stability margins of  $(\delta_1, \delta_2)$  on the two channels where  $\delta_1 \geq \delta_1^0$  and  $\delta_2 \geq \delta_2^0$ . Using ad hoc techniques it can be shown that for  $P_0$  as above there exist perturbation pairs  $(\Delta_1, \Delta_2)$  of norms  $(0.25, 0)$ ;  $(0.167, 0.05)$  and  $(0.08, 0.25)$  such that  $P_0 + \text{diag}(\Delta_1, \Delta_2)$  has one less unstable pole than  $P_0$ . Also, an (exhaustive!) search reveals that these are the "smallest" size diagonal perturbations that decrease the number of unstable poles of  $P_0$  (in terms of the partial ordering on  $\mathbb{R}_+^2$ ). Thus, one may conclude that for desired stability margins above the curve  $\beta$  (shown in the dotted line) there do not exist any controllers nonlinear and/or time varying that achieve these margins. However, for desired stability margins between the curves  $\alpha$  and  $\beta$  while LTI controllers are useless, it may be the case that suitable nonlinear control provides these margins. Thus unlike the case of completely unstructured modeling uncertainty (see Khargonekar et al. [1987] and Poolla and Ting [1987]) the possible benefit of nonlinear compensation cannot as yet be ruled out for block-structured problems. This interesting open issue is currently under investigation.

**Example (5.5).** Consider again an additive structured family of plant models as in (2.3):

$$F_{\delta} = \left\{ \begin{bmatrix} \frac{1}{s-3} + \frac{1}{s-2} & \frac{15}{s-3} \\ \frac{15}{s-2} & \frac{1}{s+1} \end{bmatrix} + \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \|\Delta_i\| \leq \delta \right\}.$$

It follows from Theorem (5.2) that the maximal stability margin  $R$  for the family above is infinite. Suppose the objective is to construct a controller  $K(s)$  that yields a stability margin of 10 using the methods of Section 5.2. Some straightforward calculations yield the following:

Step III:  $\gamma = 10$ ,  $R_1 = I_2$  and  $R_2 = \text{diag}(4, 0.25)$ .

Step IV: The interpolation problem involved is to find an outer function  $d(s)$  such that  $d(2) = 4$  and  $d(3) = .25$ . The associated Nevanlinna-Pick matrix is

$$N_m = \begin{bmatrix} \frac{\frac{1}{4^m}}{2} & \frac{\frac{1}{4^m} + (.25)^m}{5} \\ \frac{\frac{1}{4^m} + (.25)^m}{5} & \frac{(.25)^m}{3} \end{bmatrix},$$

and it can be verified that the smallest integer  $m$  for which  $N_m \geq 0$  is  $m = 7$ . Thus, one can determine a 14th-order outer interpolant via (Krein and Nudelman [1977]) to be

$$d(s) = \left( \frac{.6s^2 + 322.8s + 326.2}{130.2s^2 + 134.8s + 8.6} \right)^7.$$

Steps V and VI: The desired controller can now be obtained as in Section 5.2 through tedious but otherwise straightforward calculation.

## CHAPTER 6

### SUBOPTIMAL SOLUTIONS FOR MULTIDISK PROBLEMS

Recently, the single objective  $H_\infty$ -optimization problem introduced by Zames [1981] as an alternative to the classical Wiener-Hopf approach to feedback synthesis has received much attention. Several interesting and important control problems, however, involve the solution of a **multiobjective**  $H_\infty$ -optimization problem, also referred to as a **multidisk** problem (see Francis and Doyle [1987]). For example, the problems of

Robust stabilization with optimal nominal disturbance rejection (6.1)

Robust simultaneous stabilization (6.2)

Optimal nominal disturbance rejection with robust stability around a failure operating point (6.3)

all reduce to (perhaps nonlinear) multidisk problems.

In complete generality these multidisk problems are very difficult and no analytic solution is known. This chapter shall focus instead on linear two-disk problems (of which (6.1) is an example). An algorithm will be presented which allows computation of an upper bound for linear two-disk problems and also a (suboptimal) controller which achieves this bound. This algorithm readily generalizes to incorporate multidisk problems.

This chapter is organized as follows. Section 6.1 contains some motivating examples and provides some background information about two-disk problems. Section 6.2 contains the main results of this chapter: a systematic algorithm for computing an absolute upper bound for a linear two-disk problem and a (suboptimal) controller which



achieves this bound. These results will be demonstrated in Section 6.3 through a few numerical examples.

### 6.1. Background Information on Multi-Disk Problems

In Chapter 2 two fundamental problems in frequency domain control were introduced: the robust stabilization problem and the uniformly or  $H_\infty$ -optimal control problem. Both of these problems involve a single performance criterion and, in the context of LTI controllers, both problems reduce to the same mathematical problem, a one-disk problem. In practice, however, a control system designer must consider additional performance criteria, i.e., one may wish to simultaneously consider both the robust stability properties and the disturbance rejection capabilities of a particular control design. For example, consider the system shown in Figure 6.4 where  $P$  belongs to a family of plants  $\mathbf{F}_\delta$  as described by (2.2).

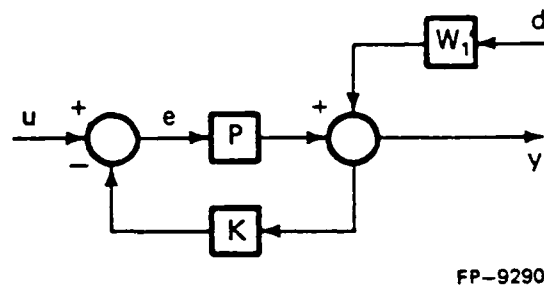


Figure 6.1. Feedback system for the robust stabilization and uniformly optimal control problem.

The objective is to design a controller  $K$  which optimally rejects disturbances  $d$  for the nominal plant  $P_0$  subject to the constraint that  $K$  robustly stabilize the family of plants  $F_\delta$ . This problem is equivalent to solving the constrained optimization problem

$$\min_{Q \in RH_\infty} \| \tilde{X} \tilde{D} W_1 + D Q \tilde{D} W_1 \|_\infty \text{ subject to } \| W_2 \tilde{Y} \tilde{D} - W_2 N Q \tilde{D} \|_\infty \leq 1,$$

which represents an instance of the two-disk problem.

In complete generality multidisk problems are very difficult and no analytic solution is known. For certain special classes of multidisk problems, a partial solution has been obtained by Ball and Helton [1979]. Also, Kwakernaak [1982] proposed an approximate solution of (6.1) by solving instead a closely related single objective  $H_\infty$ -optimization problem ("mixed sensitivity"). Finally, O'Young and Francis [1985] explicitly solve a special case of (6.1) under some reasonable hypothesis.

This chapter focuses on linear multidisk problems (of which (6.1) is an example). For purposes of clarity the following exposition deals only with two-disk problems; all results easily generalize to incorporate more than two competing optimization objectives. In general terms, a (linear) two-disk optimization problem may be formulated as follows: Given matrix  $H_\infty$ -functions  $A, B, C, \tilde{A}, \tilde{B}$ , and  $\tilde{C}$  of compatible dimensions, determine an  $H_\infty$  function  $Q$  that "jointly" minimizes  $\|A + BQC\|_\infty$  and  $\|\tilde{A} + \tilde{B}Q\tilde{C}\|_\infty$ . Examples of this joint minimization include

$$\min_Q \|A + BQC\|_\infty \text{ subject to } \|\tilde{A} + \tilde{B}Q\tilde{C}\|_\infty < \gamma \quad (6.4)$$

where  $\gamma$  is a prescribed number, or

$$\min_Q (\|A + BQC\|_\infty^2 + \|\tilde{A} + \tilde{B}Q\tilde{C}\|_\infty^2) \quad (6.5)$$

or, more generally,

$$\min_Q \phi( \|A + BQC\|_\infty, \| \tilde{A} + \tilde{B}Q\tilde{C} \|_\infty ) \quad (6.6)$$

$$\text{subject to } \psi( \|A + BQC\|_\infty, \| \tilde{A} + \tilde{B}Q\tilde{C} \|_\infty ) \leq 1$$

where  $\phi$  and  $\psi$  are some prescribed functions.

## 6.2. The Algorithm

In this section an algorithm is presented which allows computation of an upper bound for linear two-disk problems and also a (suboptimal) controller that achieves this bound (see also Ting and Poolla [1987]). This algorithm readily generalizes to incorporate multidisk problems. Hara and Katori [1986] have previously investigated the constrained  $H_\infty$ -optimization problem (which is an instance of the linear two-disk problem). For single-input single-output systems they derive upper and lower bounds on achievable performance bounds. However, their lower bounds are too conservative because they do not incorporate analyticity requirements, and their upper bounds are incomputable because they involve a graph metric calculation (see Vidyasagar [1984]). This section describes a computable upper bound for multiinput multioutput two-disk problems which requires only the solution of several one-disk problems.

To begin, consider the following

**Definition (6.1):** Let  $X$  be a vector space and let  $f: X \rightarrow \mathbb{R}^+$  and  $G: X \rightarrow \mathbb{R}^+$  be nonnegative functionals on  $X$ . Consider the problem of jointly minimizing (in some sense)  $f$  and  $g$  over  $X$ . A point  $x^*$  in  $X$  is called **Pareto-optimal** if there does not exist any  $x$  in  $X$  such that

$$f(x) \leq f(x^*) \quad \text{and} \quad g(x) \leq g(x^*)$$

with at least one of the two inequalities being strict

The objective of this definition is to eliminate clearly suboptimal solutions to the problem of jointly minimizing  $f$  and  $g$ . The following result (see DaCunha and Polak [1967] or Vincent and Grantham [1981]) characterizes the set of Pareto-optimal solutions.

**Lemma (6.1):** Let  $X$  be a convex set, and let  $f: X \rightarrow \mathbb{R}^+$ ,  $g: X \rightarrow \mathbb{R}^+$  be convex nonnegative functionals. Then,  $x^*$  in  $X$  is Pareto-optimal if and only if there exists  $\alpha \in [0, 1]$  such that  $x^*$  minimizes  $\alpha f(x) + (1-\alpha)g(x)$  over  $X$ .

□

Consider now the linear two-disk problem (6.4). Observing that  $H_\infty$  is a convex set and that  $f(Q) = \|A+BQC\|_\infty$  and  $g(Q) = \|\hat{A} + \hat{B}Q\hat{C}\|_\infty$  are convex nonnegative functionals on  $H_\infty$  (via the triangle inequality), it follows that Lemma (6.1) applies. Therefore, the problem of jointly minimizing  $f$  and  $g$  can be solved exactly if the set of all Pareto-optimal  $Q$  can be computed. For instance, consider problem (6.4). For each  $\alpha$  in  $[0,1]$  suppose it is possible to compute

$$\arg \inf_Q [\alpha f(Q) + (1-\alpha)g(Q)] \stackrel{\Delta}{=} Q^*(\alpha). \quad (6.7)$$

Suppose  $f(Q^*(\alpha))$  and  $g(Q^*(\alpha))$  are plotted as functions of  $\alpha$  as in Figure 6.2. It is clear that  $g$  is a monotone increasing function of  $\alpha$  and  $f$  is monotone decreasing. The constraint  $g(Q) \leq \gamma$  forces one to consider only Pareto-optimal  $Q$  corresponding to  $\alpha \in [0, \alpha_0]$  and it is now clear that an optimal solution to this problem is  $Q_{\text{opt}} = Q^*(\alpha_0)$ .

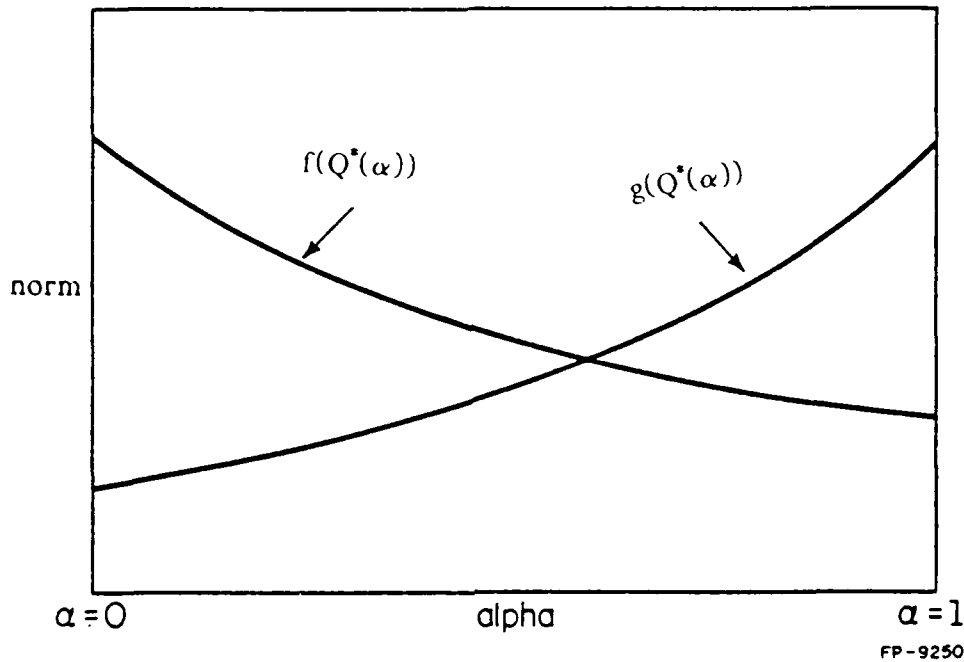


Figure 6.2. Plots of  $f(Q^*(\alpha))$  and  $g(Q^*(\alpha))$ .

The essential difficulty in using this technique to solve linear two-disk problems is computing  $Q^*(\alpha)$  as in (6.7), in order to obtain the plots in Figure 6.2. The following is a step-by-step description of an algorithm which provides a (suboptimal) approximation of the plots of Figure 6.2. In the following section an example will be presented which illustrates this algorithm.

Consider the general linear two-disk problem (6.6).

**STEP I:** Compute

$$\begin{aligned} Q^*(0) &\stackrel{\Delta}{=} \arg \inf_{Q \in H_{\infty}} \| \tilde{A} + \tilde{B}Q\tilde{C} \|_{\infty} \\ Q^*(1) &\stackrel{\Delta}{=} \arg \inf_{Q \in H_{\infty}} \| A + BQC \|_{\infty}. \end{aligned} \quad (6.8)$$

**Remark (6.1):** This step involves only the solution of two one-disk problems, which result from considering each of the two competing objectives individually. Efficient

computational methods exist for finding solutions to these problems based on the theory of Hankel operators (see Chu and Doyle [1985], Verma [1985], and Glover [1984]). While the solution is **not unique**, any infemizing choice of  $Q^*$  will suffice.

□

STEP II: For  $\alpha$  in  $[0,1]$ , define

$$\tilde{Q}(\alpha) \stackrel{\Delta}{=} \alpha Q^*(1) + (1 - \alpha) Q^*(0) \quad (6.9)$$

and plot  $\tilde{f}(\alpha) \stackrel{\Delta}{=} \|A + B\tilde{Q}(\alpha)C\|_{\infty}$  and  $\tilde{g}(\alpha) \stackrel{\Delta}{=} \|\tilde{A} + \tilde{B}\tilde{Q}(\alpha)\tilde{C}\|_{\infty}$  versus  $\alpha$ .

Remark (6.2):  $\tilde{f}$  and  $\tilde{g}$  serve as suboptimal approximations for  $f$  and  $g$ . it readily follows from the triangle inequality that  $\tilde{f}$  and  $\tilde{g}$  are both convex functions of  $\alpha$ .

□

STEP III: Determine the set

$$\Omega = \{\alpha : \psi(\tilde{f}(\alpha), \tilde{g}(\alpha)) \leq 1\}$$

Remark (6.3): In this step one may use these approximations to determine a "feasible set"  $\Omega$  of  $\alpha$ 's meeting the constraints.

□

STEP IV: Compute (graphically)

$$\alpha_0 \stackrel{\Delta}{=} \arg \inf_{\alpha \in \Omega} \phi(\tilde{f}(\alpha), \tilde{g}(\alpha)).$$

Then, a (suboptimal) upper bound solution to the linear 2-disk problem (6.6) is given by

$$Q_0 = \tilde{Q}(\alpha_0).$$

Remark (6.4). This algorithm yields an upper bound  $\alpha_0$  because the choice  $Q_0$  achieves  $\phi(\tilde{f}(\alpha), \tilde{g}(\alpha)) = \alpha$ .

□

### 6.3 A Numerical Example

In this section the algorithm described in the previous section will be demonstrated via an example. Consider the family of plants (as in Doyle and Stein [1981])

$$\mathbf{F}_\delta = \{P_0(I + \Delta W_2) : \|\Delta\| < 1\}$$

where  $P_0(s) = \frac{s-2}{s-12}$ ,  $W_2(s) = \frac{1}{3} \left[ \frac{s+1}{s+2} \right] \left[ \frac{(s+6)^2}{s^2+2s+37} \right]$  and all plants in  $\mathbf{F}_\delta$  are constrained

to have exactly one unstable pole. The design objective is to uniformly optimally control the nominal plant  $P_0$  whilst requiring robust stability. More specifically, the objective is to design a controller  $K(s)$  that

- i. internally stabilizes every plant in  $\mathbf{F}_\delta$ , and
- ii. infemizes  $\|W_1(I+P_0K)^{-1}\|_\infty$  where  $W_1(s) = \frac{1}{30} \left( \frac{s+6}{s+1} \right)$ .

First introduce (as in Desoer et al. [1980]) stably coprime factorizations of  $P_0$  as  $P_0 = ND^{-1}$  with  $XN + YD = I$ . Using standard methods one obtains

$$N = \left[ \frac{s-2}{s+6} \right] \quad D = \left[ \frac{s-12}{s+6} \right] \quad X = 1.8, \quad Y = -0.8. \quad (6.10)$$

Following Theorem (3.1) the set of all controllers that stabilize  $P_0$  can be parameterized as

$$K(s) = (X + DQ)(Y - NQ)^{-1}, \quad Q \in H_\infty. \quad (6.11)$$

From the well-known results of Zames and Francis [1983] and Doyle and Stein [1981], the design problem reduces to

$$\inf_{K = \text{stabilizing } P_0} \|W_1(I + P_0K)^{-1}\|_\infty \quad \text{subject to} \quad \|W_2P_0K(I + P_0K)^{-1}\|_\infty \leq 1. \quad (6.12)$$

Substituting (6.11) above yields

$$\inf_{Q \in H^p} \|W_1 YD - W_1 NQD\|_\infty \text{ subject to } \|W_2 NX + W_2 NQD\|_\infty \leq 1, \quad (6.13)$$

which is an instance of the linear two disk problem (6.6) with

$$A = W_1 YD, \quad B = -W_1 N, \quad C = D$$

$$\tilde{A} = W_2 NX, \quad \tilde{B} = W_2 N, \quad \tilde{C} = D$$

$$\phi = \|A + BQC\|_\infty, \quad \psi = \|\tilde{A} + \tilde{B}Q\tilde{C}\|_\infty.$$

The algorithm described in Section 6.2 may now be employed to obtain a suboptimal controller for this feedback synthesis problem.

**STEP I:** Using standard methods the following solutions to the one-disk problems described in (6.8) may be obtained

$$Q^*(0) = 0.2547 \left[ \frac{s-21.3}{s+1} \right], \quad Q^*(1) = 2.933 \left[ \frac{s+6}{s+12} \right].$$

**STEP II:** One may now plot  $\tilde{f}(\alpha)$  and  $\tilde{g}(\alpha)$  versus  $\alpha$ . Numerically this results in the following plot of Figure 6.3.



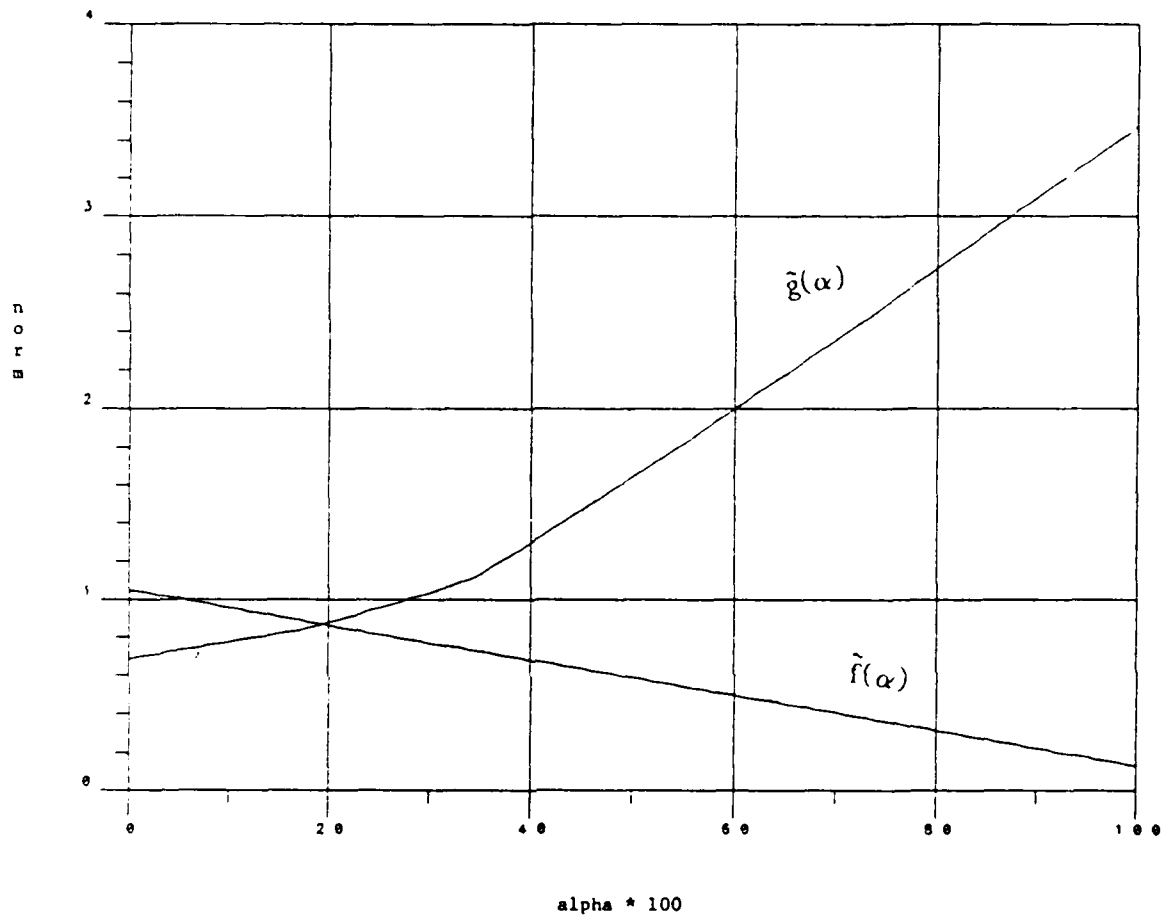


Figure 6.3. Plots of  $\tilde{f}(\alpha)$  and  $\tilde{g}(\alpha)$ .

STEP III: It is now clear that the set  $\Omega$  is

$$\Omega = [0, 0.265].$$

STEP IV:  $\alpha_0 = \arg \inf_{\alpha \in \Omega} \|A + BQC\|_{\infty} = 0.265$  and

$$Q_0 = .1872 \left[ \frac{s-21.3}{s+1} \right] + 0.7772 \left[ \frac{s+6}{s+12} \right].$$

Finally, one obtains  $\|A + BQ_0C\|_{\infty} = 0.7998$ . The corresponding controller can be determined via (6.11) to be

$$W(s) = \left[ 1.8 + \left[ \frac{s-12}{s+6} \right] Q_0 \right] \left[ -.8 - \left[ \frac{s-2}{s+6} \right] Q_0 \right]^{-1}.$$

Remark (6.5): It is interesting to compare the upper bound for the linear two-disk problem with the results achievable when considering each of the two objectives individually. For example, if one were concerned only with the problem of uniformly optimal control of  $P_0$  without any regard for the robust stability of  $\mathbb{F}_\delta$ , then one could reduce the  $H_{\infty}$ -norm of the weighted sensitivity function to 0.1244. Conversely, if one focuses solely upon finding the optimal robustly stabilizing controller, then the  $H^{\infty}$ -norm of the resulting weighted sensitivity function rises to 1.043. One of the main features of the curve shown in Figure 6.3, aside from providing an upper bound for linear two-disk problems, is to graphically display the design tradeoffs inherent in multi-objective problems.

□

## CHAPTER 7

## CONCLUDING REMARKS

This dissertation has detailed the results of an investigation into some fundamental issues in control system design. The results obtained here are vital in answering some basic questions regarding what can and cannot be accomplished using feedback control. This chapter provides a brief summary of the main results and contains some directions for possible future research.

In Chapter 4 it was demonstrated that as far as the problem of robustly stabilizing a dynamic family of plants is concerned, the best controllers are linear and time-invariant. In particular, this implies that adaptive control laws (note: these generally satisfy the technical hypothesis of Theorem (4.5)) offer no advantage over LTI controllers in the context of robustly stabilizing a purely dynamic family of plants. Due to the proliferation of available controller synthesis methods, results of this type are invaluable in ruling out various synthesis methods for certain classes of control problems.

Adaptive control, however, intends primarily to deal with parametric modeling uncertainty. In this context, an interesting result due to Martensson [1985] demonstrates that any parametric family of plant models with compact parameter variation can be robustly stabilized by using some nonlinear time-varying controller. Martensson's treatment, however, deals with internal (state) stability and cannot readily be incorporated into the input-output framework described in this dissertation. It is nevertheless apparent that for robust stabilization problems involving purely parametric uncertainty, nonlinear feedback is far superior to linear time-invariant feedback. Blending this result with those of Chapter 4 one is led to conjecture the following design

principle:

The precision to which a plant model is known determines what type of controller is best suited for controlling the plant.

More precisely, it appears that linear time-invariant controllers are optimal for controlling families of plants featuring primarily dynamic uncertainty. Conversely, it appears that families of plants featuring primarily parametric uncertainty are best controlled by nonlinear time-varying (or adaptive) controllers. Intuitively, this follows from the belief that families of plants featuring primarily parametric uncertainty could possibly be more accurately identified through further input-output experiments. However, for families of plants featuring primarily dynamic uncertainty, further information about the "true" plant model cannot be obtained.

A representation of modeling uncertainty that combines both dynamic and parametric uncertainty is the notion of mixed uncertainty as described in Chapter 2. This representation provides a setting to attempt to quantify the ideas alluded to in this design principle. In other words, one would like to determine how much dynamic uncertainty can be exhibited by a family of plant models before nonlinear time-varying controllers no longer provide significantly better performance than linear time-invariant controllers. Similarly, one would also like to determine how much parametric uncertainty can be contained in a family of plant models before nonlinear time-varying controllers become far superior to linear time-invariant controllers. Recently, (see, for example, Rohrs et al. [1982]) there has been interest in examining how adaptive control laws perform with unmodeled dynamics. One possible representation of this modeling error is described by a family of plants featuring mixed uncertainty as given in (2.4), i.e.,  $P_\delta$  represents a combination of parametric and dynamic uncertainty. An interesting open

question then, is determining an exact numerical characterization of the largest radius  $\delta_{\max}$  of unmodeled dynamics that can possibly be stabilized using any NLTV controller.

A closely related problem involves the construction of an explicit adaptive or switching control scheme for a family of plants featuring mixed uncertainty. More precisely, consider a single plant  $P_0$  contained in a family of plants  $F_\delta$  as described in (2.4). Suppose that for each fixed  $\alpha \in \Omega$  there exists a linear time-invariant controller  $K_\alpha$  that robustly stabilizes the resulting family of plants. The key question then is: Does there exist some method of gain-scheduling or switching between the individual controllers  $K_\alpha$  which ensures that the resulting closed-loop system is stable? Along these lines, Fu and Barmish [1986] have recently applied the idea of switching control to the problem of robustly stabilizing a family of plants with compact parameter variations. They exploit the fact that any stabilizing controller for a given nominal plant model contains some amount of robustness to develop a switching law between a finite number of controllers which guarantees stability for the true plant model. Their results are not directly applicable to the frequency domain robust stabilization problem described above because they deal with exponential state stability as opposed to input-output stability. Such problems are of interest to the adaptive control community because they involve a method for adaptive stabilization without restrictive assumptions such as minimum-phase properties.

In Chapter 5 a new iterative procedure was introduced for synthesizing robustly stabilizing controllers for families of plants featuring block-structured uncertainties. This method is referred to as residue iteration and provides a computationally attractive alternative to the established  $\Lambda$ - $Q$  iteration methods. In addition, this approach was helpful in providing valuable insight into the problem and triggered the development of

some sufficient conditions on the nominal plant to ensure infinite stability margins. To facilitate the computations involved in performing residue iteration, a new method was introduced for finding low-order outer interpolating functions. This new procedure is also less computationally intensive and provides lower-order outer functions than previously established methods.

Certain numerical aspects of the residue iteration procedure deserve further attention. Although residue iteration is numerically attractive as compared to  $\Lambda$ - $Q$  iteration, both procedures share a common difficulty of requiring the solution of a non-convex optimization problem. Recall that (5.1) is not jointly convex in  $\Lambda$  and  $Q$  and (5.7) is not concave in the residues  $R_i$ . Thus, neither iterative procedure can guarantee convergence to the optimal robustly stabilizing controller. It may be possible to show, however, that the functional  $\underline{\lambda}^{1/2}(M(R)W(R))$  is quasi-concave in the residues  $R_i$ . This would be desirable because quasi-concave functions have at most one local maximum. Thus, upon arrival at a local maximum in (5.7) it is only necessary to check the boundary behavior of  $\underline{\lambda}^{1/2}(M(R)W(R))$  as  $R$  approaches  $\infty$  to determine the global maximum.

The development of residue iteration raises several related issues regarding this robust stabilization problem. For example, one would like to investigate whether it is possible to extend this controller synthesis procedure to arbitrary (non-square) families of plants featuring block-structured uncertainty. In addition, the utility of nonlinear time-varying controllers for robustly stabilizing families of plants featuring block-structured uncertainty is a completely open question.

Chapter 6 detailed a step-by-step algorithm which allows for computation of an upper bound for multiobjective  $H_\infty$ -optimization problems. These results are obtained graphically and require only the solution of several single-objective  $H_\infty$ -optimization

problems. The graphs obtained are useful in demonstrating the design tradeoffs inherent in multiobjective problems.

## APPENDIX

## PROOF OF THEOREM (4.3)

Throughout this appendix,  $M$  is a (fixed),  $m$ -input,  $p$ -output NLTV operator of the form  $M = A + BQC$ , where  $A$ ,  $B$ , and  $C$  are causal, stable, linear time-invariant operators with  $C$  being  $r \times r$ , full rank and  $Q$  is some (fixed) causal incrementally stable NLTV operator. Introduce inner-outer and outer-inner factorizations of  $B$  and  $C$ , respectively (see Averson [1975]) as

$$B = B_i B_o, \quad C = C_o C_i.$$

Further, assume that

$$\|B_i^* A C_i^*\|_H > 1 + \epsilon, \quad (\text{A.1})$$

where the subscript  $H$  denotes the Hankel norm (see Chapter 3).

In order to establish the proof of Theorem (4.3) the following intermediate results shall be required.

**Lemma (A.1).** Let  $T$  be any fixed integer. Then, there exists an input sequence  $w$  in  $h_2^m$  (dependent on  $T$ ) of duration  $K-T$  and such that

$$\frac{1}{2} \leq \|w\| \leq 1, \quad P_T(w) = 0, \quad \frac{\|P_K M w\|}{\|P_K w\|} > 1 + \epsilon. \quad (\text{A.2})$$

**Proof.** First note that from Theorem (4.2),

$$\inf_{Q_1: \text{causal, stable NLTV}} \|A + BQ_1 C\|_{bh_2} = \|B_i^* A C_i^*\|_H > 1 + \epsilon. \quad (\text{A.3})$$

To verify (A.2), define a causal, stable NLTV operator  $Q_2 := z^T Q z^{-T}$  where  $z^{-1}$  is the unit delay operator. It now follows from (A.3) that there exist some  $\lambda$  in  $bh_2^m$  such that



$\|(A + BQ_2C)\lambda\| / \|\lambda\| > 1 + \epsilon$ . Define  $v = z^{-T}\lambda$ . Then,

$$\begin{aligned} \frac{\|Mv\|}{\|v\|} &= \frac{\|(A + BQC)z^{-T}\lambda\|}{\|\lambda\|} = \frac{\|z^T(A + BQC)z^{-T}\lambda\|}{\|\lambda\|} \\ &= \frac{\|(A + BQ_2C)\lambda\|}{\|\lambda\|} > 1 + \epsilon. \end{aligned}$$

Since  $v$  is in  $h_2^m$ , it follows that there exist integers  $T_1$  and  $T_2$  such that

$$\begin{aligned} \frac{1}{2} &\leq \|P_N v\| \leq 1, \quad \text{all } N \geq T_1 \\ \frac{\|P_N(A + BQC)v\|}{\|P_N v\|} &> 1 + \epsilon, \quad \text{all } N \geq T_2. \end{aligned}$$

Let  $K = \max\{T_1, T_2\}$  and define  $w = P_K(v)$ . It is now easy to see that  $w$  satisfies (A.2), proving the claim. □

**Proposition (A.1).** Suppose there exists some input sequence  $w^1$  of duration  $K_1$  such that

$$\frac{\|P_{K_1} M(w^1, *, *, \dots)\|}{\|P_{K_1}(w^1, *, *, \dots)\|} > 1 + \epsilon.$$

Then, there exists an extension  $w^2$  of duration  $K_2 - K_1$  such that

$$\frac{1}{2} \leq \|w^2\| \leq 1, \quad \frac{\|P_{K_2} M(w^1, w^2, *, *, \dots)\|}{\|P_{K_2}(w^1, w^2, *, *, \dots)\|} > 1 + \epsilon. \quad (\text{A.4})$$

**Proof.** Given any sequence  $v$  in  $h_{2\epsilon}^m$  and integers  $K_1, K_2$  with  $K_1 < K_2$ , define the finite sequence

$$\{v\}_{[K_1, K_2)} = (v_{K_1}, v_{K_1+1}, \dots, v_{K_2-2}, v_{K_2-1}).$$

Let  $\delta > 0$  be any (fixed) real number. Since  $w^1$  is in  $h_2$  and since  $M$  is stable, it follows

that there exists an integer  $N$  (without loss of generality  $N > K_1$ ) such that  $\|(1 - P_N)M(w^1, 0, 0, \dots)\| < \delta$ .

Let  $\lambda = C(w^1, 0, 0, \dots)$  and define a nonlinear operator  $\hat{Q}$  by

$$\hat{Q}(u) = Q(u + \lambda) - Q(\lambda). \quad (A.5)$$

It is easy to verify that (A.5) defines a causal operator. Further, since  $Q$  is incrementally stable, it follows that

$$\|\hat{Q}\| = \sup_u \frac{\|Q(u + \lambda) - Q(\lambda)\|}{\|u\|} \leq \|Q\|_{inc} < \infty.$$

Thus,  $\hat{Q}$  is stable. Define  $\hat{M} = A + B\hat{Q}C$ . With  $T = N$  in Lemma (A.1) it follows that there exists an input sequence  $v$  of duration  $K_2$  such that

$$\frac{1}{2} \leq \|v\| \leq 1, \quad \|P_N v\| = 0, \quad \frac{\|P_{K_2} \hat{M} v\|}{\|P_{K_2} v\|} > 1 + \epsilon.$$

Let  $w^2 = [v]_{[K_1, K_2]}$ , and notice that  $\frac{1}{2} \leq \|w^2\| \leq 1$ . Notice now that

$$\begin{aligned} \|P_{K_2} M(w^1, w^2, *, *, \dots)\|^2 &= \|P_N M(w^1, *, *, \dots)\|^2 + \|(1 - P_N)M(w^1, w^2, *, *, \dots)\|^2 \\ &> (1 + \epsilon)^2 \|w^1\|^2 + \|(P_{K_2} - P_N)M(w^1, w^2, *, *, \dots)\|^2. \end{aligned} \quad (A.6)$$

Also,

$$\begin{aligned} &\|(P_{K_2} - P_N)M(w^1, w^2, *, *, \dots)\| \\ &\geq \|P_{K_2} M(w^1, w^2, *, *, \dots) - P_{K_2} M(w^1, 0, 0, \dots)\| \\ &= \|(P_{K_2} - P_N)M(w^1, 0, 0, \dots)\| \end{aligned}$$

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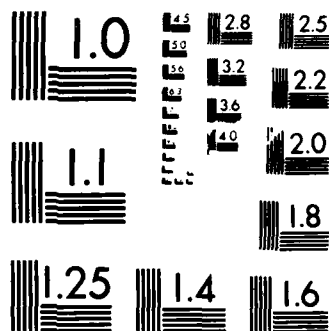
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$$\geq \|P_{K_2} \hat{M}v\| - \delta \geq (1 + \epsilon) \|v\| - \delta = (1 + \epsilon) \|w^2\| - \delta.$$

Combining (A.6) and (A.7) yields

$$\|P_{K_2} M(w^1, w^2, *, *, \dots)\|^2 > (1 + \epsilon)^2 (\|w^1\|^2 + \|w^2\|^2) - 2\delta(1 + \epsilon) \|w^2\|.$$

Since  $\delta$  can be chosen to be arbitrarily small and since  $\|w^2\| \leq 1$ , it follows that there exist some  $w^2$  such that

$$\|P_{K_2} M(w^1, w^2, *, *, \dots)\| > (1 + \epsilon) \|P_{K_2}(w^1, w^2, *, *, \dots)\|,$$

proving (A.4). □

One is now in a position to prove the following

**Theorem (4.3).** There exists an input sequence  $u$  in  $h_{2e}^m$  such that

$$\lim_{N \rightarrow \infty} \|P_N u\| = \infty \quad (4.8)$$

$$\lim_{N \rightarrow \infty} \frac{\|P_N M u\|}{\|P_N u\|} > 1 + \epsilon. \quad (4.9)$$

**Proof.** With  $T = 0$  in Lemma (A.1) it follows that there exists some input sequence  $w^1$  of duration  $K_1$  such that

$$\frac{1}{2} \leq \|w^1\| \leq 1, \quad \frac{\|P_{K_1} M(w^1, *, *, \dots)\|}{\|P_{K_1}(w^1, *, *, \dots)\|} > 1 + \epsilon. \quad (A.8)$$

Now, from Proposition (A.1), one may conclude that there exists an extension  $w^2$  of duration  $K_2 - K_1$  such that

$$\frac{1}{2} \leq \|w^2\| \leq 1, \quad \frac{\|P_{K_2} M(w^1, w^2, *, *, \dots)\|}{\|P_{K_2}(w^1, w^2, *, *, \dots)\|} > 1 + \epsilon. \quad (\text{A.9})$$

Continuing in this fashion, it is possible to determine using Proposition (A.1) that there exists a sequence  $u = (w^1, w^2, w^3, \dots)$  where  $w^i$  is of duration  $K_i - K_{i-1}$ , with the following properties:

$$\frac{1}{2} \leq \|w^i\| \leq 1, \quad \frac{\|P_{K_i} Mu\|}{\|P_{K_i} u\|} > 1 + \epsilon, \quad \text{for all } i. \quad (\text{A.10})$$

Now let  $N$  be any integer, say  $K_i < N \leq K_{i+1}$ . Then,

$$\frac{\|P_N Mu\|}{\|P_N u\|} \geq \frac{\|P_{K_i} Mu\|}{\|P_{K_{i+1}} u\|} > (1 + \epsilon) \frac{\|P_{K_i} u\|}{\|P_{K_{i+1}} u\|} \geq (1 + \epsilon) \left[ \frac{i}{i+2} \right]^{\frac{1}{2}}.$$

Consequently,  $\lim_{N \rightarrow \infty} \frac{\|P_N Mu\|}{\|P_N u\|} > 1 + \epsilon$ , proving (4.10), and clearly  $u$  is unbounded,

proving (4.9). This completes the proof.

□

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